Midterm 2

Friday, Feburary 28, 2020

Let $X \subset \mathbb{R}^2$ be a connected, 1-dimensional manifold. You will prove that for almost every point $q \in \mathbb{R}^2$, the map $X \to \mathbb{R}$ given by $p \mapsto |p - q|^2$ is a Morse function. The same is true in higher dimensions, but the proof is harder.

You are given a surjective local diffeomorphism $\gamma \colon \mathbb{R} \to X$, which you can write in components as

$$\gamma(t) = (x(t), y(t)).$$

1. (5 points) Fix a point $(a, b) \in \mathbb{R}^2$, and let $f \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = (x(t) - a)^2 + (y(t) - b)^2,$$

   which is the square of the distance from $\gamma(t)$ to (a, b).

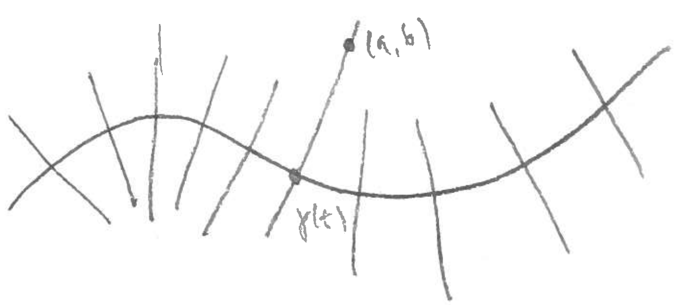
   Compute $f'(t)$.

2. (10 points) Let $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(t, u) = (x(t) + uy'(t), y(t) - ux'(t)).$$

   Show that $f'(t) = 0$ if and only if there is a $u \in \mathbb{R}$ with $F(t, u) = (a, b)$.

   (Geometric remark: If t is fixed and u varies then $F(t, u)$ traces out the line that's perpendicular to the curve at $\gamma(t)$. So you are proving that $f'(t) = 0$ if and only if this line passes through (a, b).)



3. (5 points) Compute $f''(t)$.

4. (5 points) Compute $dF_{(t,u)}$.

5. (10 points) Suppose that $f'(t) = 0$, so by problem 2 there is a $u \in \mathbb{R}$ with $F(t, u) = (a, b)$.

   Show that (t, u) is a critical point of F if and only if $f''(t) = 0$.

6. (5 points) Define what it means for f to be a Morse function.

   Hint: Because f is a function of one variable, you can simplify the the definition a lot.

7. (5 points) State Sard's theorem for the map F.

8. (5 points) Now let (a, b) vary, and consider the set of points $(a, b) \in \mathbb{R}^2$ for which the function f defined in problem 1 is *not* a Morse function. Show that this set has measure zero.

   Hint: Don't work hard, just use what you've already done.