# Final Exam

Tuesday, December 6, 2016

Each part of each problem is worth 5 points, for a total of 65 points.

1. Let $\varphi\colon R \to S$ be a homomorphism, let $I \subset R$ be an ideal, and let

$$J = \varphi(I) = \{j \in S : j = \varphi(i) \text{ for some } i \in I\}.$$

   (a) Show that if $\varphi$ is surjective then $J$ is an ideal.

   (b) Give an example to show that if $\varphi$ is not surjective then $J$ need not be an ideal.

2. Let $S \subset \mathbb{Q}$ be the set of numbers of the form $m/2^n$, where $m \in \mathbb{Z}$ and $n \in \{0, 1, 2, \dots\}$.

   (a) Show that $S$ is a subring of $\mathbb{Q}$.

   (b) Show that $S/5 \cong \mathbb{Z}/5$.
   Hint: Of course you want a homomorphism $\varphi\colon S \to \mathbb{Z}/5$ with $\ker\varphi = (5)$; the main question is where to send $\frac{1}{2}$. Observe that $2 \cdot 3 = 1$ in $\mathbb{Z}/5$.

   (c) Let $f \in \mathbb{Z}[x]$. Show that if $f(\frac{1}{2}) = 0$ then $2x - 1 \mid f$.

   (d) Show that $S \cong \mathbb{Z}[x]/(2x-1)$.
   Hint: Consider an appropriate homomorphism $\varphi\colon \mathbb{Z}[x] \to \mathbb{Q}$.

(Continued on back.)

3. Recall that $\mathbb{Q}[[x]]$ is the ring of formal power series

$$f = a_0 + a_1 x + a_2 x^2 + \cdots$$

where $a_i \in \mathbb{Q}$, and infinitely many $a_i$ may be non-zero. It does not make sense to talk about the *degree* of $f$, but for $f \neq 0$ we define the *order* of $f$ to be the smallest $i$ such that $a_i \neq 0$. For example, $\operatorname{ord}(1 + x + x^2) = 0$ and $\operatorname{ord}(x + x^2) = 1$.

(a) Show that $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$.

(b) Let

$$f = a_0 + a_1 x + a_2 x^2 + \cdots \in \mathbb{Q}[[x]].$$

Show that if $f$ is a unit then $a_0 \neq 0$.

(c) Show that if $a_0 = 1$ then $f$ is a unit.

Hint: Think about the geometric series:

$$\frac{1}{1 + \text{stuff}} = 1 - \text{stuff} + \text{stuff}^2 - \text{stuff}^3 + \cdots .$$

But make sure that your argument doesn't accidentally prove that $1 + (x - 1)$ is a unit.

(d) Show more generally that if $a_0 \neq 0$ then $f$ is a unit.

Hint: Factor $a_0$ out of $f$, and note that the product of two units is a unit.

(e) Let $f \in \mathbb{Q}[[x]]$ be non-zero, and let $n = \operatorname{ord}(f)$. Show that the ideal $(f) = (x^n)$.

Hint: Factor $x^n$ out of $f$; then what's left is a unit.

(f) Show that every non-zero ideal $I \subset \mathbb{Q}[[x]]$ is of the form $(x^n)$ for some $n$.

Hint: Choose an $f \in I$ of minimal order. Do not assume that $I$ is finitely generated.

Remarks: Thus we have proved that $\mathbb{Q}[[x]]$ is a principal ideal domain, and that it has only one maximal ideal. A ring with a unique maximal ideal is called a *local ring*. A local PID is called a *discrete valuation ring*; in this example the valuation is what we've been calling the order of $f$.

(g) Which of the ideals $(0), (x), (x^2), (x^3), \ldots$ are prime ideals?