Final Exam

Tuesday, December 6, 2016

Each part of each problem is worth 5 points, for a total of 65 points.

1. Let \( \varphi: R \to S \) be a homomorphism, let \( I \subset R \) be an ideal, and let
   \[ J = \varphi(I) = \{ j \in S : j = \varphi(i) \text{ for some } i \in I \} \].
   
   (a) Show that if \( \varphi \) is surjective then \( J \) is an ideal.
   
   (b) Give an example to show that if \( \varphi \) is not surjective then \( J \) need not be an ideal.

2. Let \( S \subset \mathbb{Q} \) be the set of numbers of the form \( m/2^n \), where \( m \in \mathbb{Z} \) and \( n \in \{0, 1, 2, \ldots\} \).
   
   (a) Show that \( S \) is a subring of \( \mathbb{Q} \).
   
   (b) Show that \( S/5 \cong \mathbb{Z}/5 \).
      
      Hint: Of course you want a homomorphism \( \varphi: S \to \mathbb{Z}/5 \) with \( \ker \varphi = (5) \); the main question is where to send \( \frac{1}{2} \). Observe that \( 2 \cdot 3 = 1 \) in \( \mathbb{Z}/5 \).
   
   (c) Let \( f \in \mathbb{Z}[x] \). Show that if \( f(\frac{1}{2}) = 0 \) then \( 2x - 1 \mid f \).
   
   (d) Show that \( S \cong \mathbb{Z}[x]/(2x - 1) \).
      
      Hint: Consider an appropriate homomorphism \( \varphi: \mathbb{Z}[x] \to \mathbb{Q} \).

(Continued on back.)
3. Recall that $\mathbb{Q}[x]$ is the ring of formal power series
\[ f = a_0 + a_1 x + a_2 x^2 + \cdots \]
where $a_i \in \mathbb{Q}$, and infinitely many $a_i$ may be non-zero. It does not make sense to talk about the degree of $f$, but for $f \neq 0$ we define the order of $f$ to be the smallest $i$ such that $a_i \neq 0$. For example, \( \text{ord}(1 + x + x^2) = 0 \) and \( \text{ord}(x + x^2) = 1 \).

(a) Show that \( \text{ord}(fg) = \text{ord}(f) + \text{ord}(g) \).
(b) Let \( f = a_0 + a_1 x + a_2 x^2 + \cdots \in \mathbb{Q}[x] \).
   Show that if $f$ is a unit then $a_0 \neq 0$.
(c) Show that if $a_0 = 1$ then $f$ is a unit.
    Hint: Think about the geometric series:
    \[
    \frac{1}{1 + \text{stuff}} = 1 - \text{stuff} + \text{stuff}^2 - \text{stuff}^3 + \cdots.
    \]
    But make sure that your argument doesn’t accidentally prove that $1 + (x - 1)$ is a unit.
(d) Show more generally that if $a_0 \neq 0$ then $f$ is a unit.
    Hint: Factor $a_0$ out of $f$, and note that the product of two units is a unit.
(e) Let $f \in \mathbb{Q}[[x]]$ be non-zero, and let $n = \text{ord}(f)$. Show that the ideal $(f) = (x^n)$.
    Hint: Factor $x^n$ out of $f$; then what’s left is a unit.
(f) Show that every non-zero ideal $I \subset \mathbb{Q}[[x]]$ is of the form $(x^n)$ for some $n$.
    Hint: Choose an $f \in I$ of minimal order. Do not assume that $I$ is finitely generated.
    Remarks: Thus we have proved that $\mathbb{Q}[[x]]$ is a principal ideal domain, and that it has only one maximal ideal. A ring with a unique maximal ideal is called a local ring. A local PID is called a discrete valuation ring; in this example the valuation is what we’ve been calling the order of $f$.
(g) Which of the ideals $(0), (x), (x^2), (x^3), \ldots$ are prime ideals?