Homework 1

Due Wednesday, October 5, 2016

You may work with other students on these problems, but you must write them up yourself, in your own words. Use pencil and double psace (skip lines). If you type, use TEX, not Microsoft Word. Do not use the shorthand $\forall$ and $\exists$; you have time to write "for every" and "there is."

1. Let X be a set. Recall that:
   - An *equivalence relation* on X is a relation $\sim$ that is reflexive ($x \sim x$), symmetric (if $x \sim y$ then $y \sim x$), and transitive (if $x \sim y$ and $y \sim z$ then $x \sim z$).
   - A *partition* of X is a collection of disjoint subsets of X whose union is all of X.
   - A *surjection* from X to another set is an "onto" map.

   (a) Show that an equivalence relation on X determines a partition of X, and vice versa.

   (b) Show that an equivalence relation on X determines a surjection onto another set, and vice versa.

2. Let R be a ring. Using only the axioms, show that:

   (a) 0 is unique: if there are elements $0, 0' \in R$ such that $x + 0 = x$ and $x + 0' = x$ for all $x \in R$, then $0 = 0'$.

   (b) 1 is unique: if there are elements $1, 1' \in R$ such that $x \cdot 1 = 1 \cdot x = x$ and $x \cdot 1' = 1' \cdot x = x$ for all $x \in R$, then $1 = 1'$.

   (c) Additive inverses are unique: given $x \in R$, if there are elements $y, z \in R$ such that $x + y = 0$ and $x + z = 0$, then $y = z$.

   (d) For all $x \in R$ we have $0 \cdot x = x \cdot 0 = 0$.

   (e) For all $x \in R$ we have $-(-x) = x$.

   (f) For all $x \in R$ we have $-x = (-1) \cdot x = x \cdot (-1)$.

(g) For all $x, y \in R$ we have $-(x+y) = -x + -y$.

(h) For all $x, y \in R$ we have $(-x) \cdot y = x \cdot (-y) = -(x \cdot y)$.

(i) For all $x, y \in R$ we have $(-x) \cdot (-y) = x \cdot y$.

3. A ring R is called a *Boolean ring* if $x^2 = x$ for all $x \in R$. (We saw an example of such a ring in lecture; what was it?) Show that every Boolean ring is commutative, that is, $xy = yx$ for all $x, y \in R$. Hint: Consider $(x+y)^2$. Also notice that $-1 = 1$ in this ring.

4. Let $M_2(\mathbb{R})$ be the ring of 2×2 matrices with real entries. Let $S \subset M_2(\mathbb{R})$ be the set of matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Show that S is a subring of $M_2(\mathbb{R})$.

5. What is one question you have about last week's lectures?