Homework 3

Due Wednesday, October 19, 2016

1. Let $R \subset M_2(\mathbb{Z})$ be the set of matrices of the form
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$
   (a) Show that R is a subring of $M_2(\mathbb{Z})$.
   (b) Show (by example) that R is non-commutative.
   (c) Show that the map
   $$\varphi\colon R \to \mathbb{Z} \times \mathbb{Z}$$
   given by
   $$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto (a, c).$$
   is a homomorphism.
   (d) Describe $\operatorname{im} \varphi$ and $\ker \varphi$.

2. Let R be the ring of continuous functions on the interval $[0, 1]$, and consider the map $\varphi\colon R \to \mathbb{R}$ given by
$$\varphi(f) = \int_0^1 f(t)\, dt.$$
Which properties of a homomorphism hold for φ? Which ones fail? (Give counterexamples for the latter.)

3. Show that there is no homomorphism $\varphi\colon \mathbb{Z}/3 \to \mathbb{Z}/2$.

4. Describe all possible homomorphisms $\varphi\colon \mathbb{Z}[i] \to \mathbb{Z}/5$.
Hint: What are the possible values of $\varphi(i)$?

5. Let R be a ring and $I, J \subseteq R$ be ideals.

   (a) Show that $I \cap J$ is an ideal.

   (b) Show that
   $$I + J = \{a + b : a \in I, b \in J\}$$
   is an ideal.

   (c) Show that
   $$IJ = \{a_1 b_1 \cdots + a_n b_n : a_i \in I, b_i \in J\}$$
   is an ideal, and that $IJ \subset I \cap J$.

   (d) Given elements $x_1, \ldots, x_n \in R$, show that the ideal $(x_1, \ldots, x_n)$ is contained in I if and only if $x_1, \ldots, x_n \in I$.

   (e) Let $R = \mathbb{Z}$, $I = (4)$, and $J = (6)$. Describe $I \cap J$, $I + J$, and IJ.

6. For $z = x + iy \in \mathbb{C}$, recall that $|z|^2 = x^2 + y^2$. We have seen that $|zw|^2 = |z|^2 |w|^2$.

   (a) Observe that if $z \in \mathbb{Z}[i]$ then $|z|^2 \in \mathbb{Z}$. Use this to show that z is a unit in $\mathbb{Z}[i]$ if and only if $|z|^2 = 1$.

   (b) Find all the units in $\mathbb{Z}[i]$.

   (c) Find all the units in $\mathbb{Z}[\sqrt{-5}]$.

   (d) Let $\omega = \frac{-1+\sqrt{-3}}{2}$, and let
   $$\mathbb{Z}[\omega] = \{x + y\,\omega : x, y \in \mathbb{Z}\}.$$
   Draw some points of $\mathbb{Z}[\omega]$ in the complex plane. Hint: There are lots of triangles and/or hexagons.

   (e) Check that $\omega^2 = -\omega - 1$. Use this to show that $\mathbb{Z}[\omega]$ is a subring of $\mathbb{C}$.

   (f) Show if $z = x + y\,\omega$ then $|z|^2 = x^2 - xy + y^2$. Observe that this is an integer. Show that z is a unit in $\mathbb{Z}[\omega]$ if and only if $|z|^2 = 1$.

   (g) Find all the units in $\mathbb{Z}[\omega]$. Hint: there are 6.

7. What is one question you have about last week's lectures?