1. Let $R \subset M_2(\mathbb{Z})$ be the set of matrices of the form

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

(a) Show that $R$ is a subring of $M_2(\mathbb{Z})$.
(b) Show (by example) that $R$ is non-commutative.
(c) Show that the map

$\varphi: R \to \mathbb{Z} \times \mathbb{Z}$

given by

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto (a, c)$.

is a homomorphism.
(d) Describe $\text{im } \varphi$ and $\ker \varphi$.

2. Let $R$ be the ring of continuous functions on the interval $[0, 1]$, and consider the map $\varphi: R \to \mathbb{R}$ given by

$\varphi(f) = \int_{0}^{1} f(t) \, dt$.

Which properties of a homomorphism hold for $\varphi$? Which ones fail? (Give counterexamples for the latter.)

3. Show that there is no homomorphism $\varphi: \mathbb{Z}/3 \to \mathbb{Z}/2$.

4. Describe all possible homomorphisms $\varphi: \mathbb{Z}[i] \to \mathbb{Z}/5$.
   Hint: What are the possible values of $\varphi(i)$?
5. Let $R$ be a ring and $I, J \subseteq R$ be ideals.

(a) Show that $I \cap J$ is an ideal.

(b) Show that

$$I + J = \{a + b : a \in I, b \in J\}$$

is an ideal.

(c) Show that

$$IJ = \{a_1b_1 + \cdots + a_nb_n : a_i \in I, b_i \in J\}$$

is an ideal, and that $IJ \subseteq I \cap J$.

(d) Given elements $x_1, \ldots, x_n \in R$, show that the ideal $(x_1, \ldots, x_n)$ is contained in $I$ if and only if $x_1, \ldots, x_n \in I$.

(e) Let $R = \mathbb{Z}$, $I = (4)$, and $J = (6)$. Describe $I \cap J$, $I + J$, and $IJ$.

6. For $z = x + iy \in \mathbb{C}$, recall that $|z|^2 = x^2 + y^2$. We have seen that $|zw|^2 = |z|^2|w|^2$.

(a) Observe that if $z \in \mathbb{Z}[i]$ then $|z|^2 \in \mathbb{Z}$. Use this to show that $z$ is a unit in $\mathbb{Z}[i]$ if and only if $|z|^2 = 1$.

(b) Find all the units in $\mathbb{Z}[i]$.

(c) Find all the units in $\mathbb{Z}[\sqrt{-5}]$.

(d) Let $\omega = \frac{-1 + \sqrt{-3}}{2}$, and let

$$\mathbb{Z}[\omega] = \{x + y\omega : x, y \in \mathbb{Z}\}.$$ 

Draw some points of $\mathbb{Z}[\omega]$ in the complex plane. Hint: There are lots of triangles and/or hexagons.

(e) Check that $\omega^2 = -\omega - 1$. Use this to show that $\mathbb{Z}[\omega]$ is a subring of $\mathbb{C}$.

(f) Show if $z = x + y\omega$ then $|z|^2 = x^2 - xy + y^2$. Observe that this is an integer. Show that $z$ is a unit in $\mathbb{Z}[\omega]$ if and only if $|z|^2 = 1$.

(g) Find all the units in $\mathbb{Z}[\omega]$. Hint: there are 6.

7. What is one question you have about last week’s lectures?