Homework 4

Due Wednesday, October 26, 2016

1. Let R be a ring and $\varphi\colon \mathbb{Q} \to R$ be a homomorphism. Show that either $R = 0$ or φ is injective. Hint: $\ker\varphi$ is an ideal of $\mathbb{Q}$, and in lecture we described all of these.

2. In lecture we saw that every ideal in $\mathbb{Q}[x]$ is *principal*: that is, for every ideal $I \subset \mathbb{Q}[x]$ there is an $f \in \mathbb{Q}[x]$ such that $I = (f)$. In contrast, show that $(2, x) \subset \mathbb{Z}[x]$ is not a principal.

3. (a) Show that there is no homomorphism $\varphi\colon \mathbb{Q}[x] \to \mathbb{Z}[y]$.
   (b) Describe all homomorphisms $\psi\colon \mathbb{Z}[y] \to \mathbb{Q}[x]$.
   (c) Optional: Show that no such ψ is surjective.

4. (a) Write down two or three elements of $\mathbb{Q}[x][y]$.
   (b) For any ring R, give an isomorphism $\varphi\colon R[x][y] \to R[y][x]$.
   Optional: Prove that your map is an isomorphism.
   (c) Write down φ of your elements from part (a).

5. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a ring homomorphism.
   (a) Let $a \in \mathbb{Z} \subset \mathbb{R}$. Show that $\varphi(a) = a$.
   (b) Let $a \in \mathbb{Q} \subset \mathbb{R}$. Show that $\varphi(a) = a$.
   Hint: Write $a = p/q$, and consider $q \cdot \varphi(a)$.
   (c) Let $a \in \mathbb{R}$. Show that if $a \geq 0$ then $\varphi(a) \geq 0$.
   Hint: Consider $\varphi(\sqrt{a})$.
   (d) Let $a, b \in \mathbb{R}$. Show that if $a \leq b$ then $\varphi(a) \leq \varphi(b)$.
   (e) Now let $a \in \mathbb{R}$ be arbitrary. Show that $\varphi(a) = a$.
   Hint: Consider $\{b \in \mathbb{Q} : b \leq a\}$ and/or $\{b \in \mathbb{Q} : b \geq a\}$.
   You can use facts from analysis without proof.
   (f) In contrast, show that the map $\varphi\colon \mathbb{C} \to \mathbb{C}$ given by $\varphi(a + bi) = a - bi$ is a homomorphism.

6. What is one question you have about last week's lectures?