Homework 5

Due Wednesday, November 2, 2016

1. Recall that the *quaternions* are

$$\mathbb{H} = \{x + yi + zj + wk : x, y, z, w \in \mathbb{R}\},$$

where multiplication is determined by

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ ij = k &= -ji \\ jk = i &= -kj \\ ki = j &= -ik. \end{aligned}$$

Show that the center of $\mathbb{H}$ is just the *real* quaternions, i.e. the ones with with $y = z = w = 0$.

2. In $\mathbb{Q}[x]$, let

$$\begin{aligned} p &= 2x^7 + 7x^5 - 4x^3 + 9x - 1 \\ q &= (x-1)^4. \end{aligned}$$

Let $R = \mathbb{Q}[x]/(x^3 - 2x + 1)$.

(a) Find an $f \in \mathbb{Q}[x]$ of degree ≤ 2 such that $\bar{p} = \bar{f} \in R$.

(b) Do the same for $\bar{q}$, $\overline{p+q}$, and $\overline{pq}$.

(c) Show that R is not an integral domain. Hint: Factor $x^3 - 2x + 1$.

(d) Show that $\bar{x}$ is a unit in R.

3. (a) For $f \in \mathbb{Q}[x]$, show that $f \in (x^2 - 5x + 6)$ if and only if $f(2) = 0$ and $f(3) = 0$. Hint: By polynomial long division we can write

$$f = (x^2 - 5x + 6) \cdot q + r$$

with $\deg r \leq 1$.

   (b) Show that $\mathbb{Q}[x]/(x^2 - 1) \cong \mathbb{Q} \times \mathbb{Q}$. Hint: Consider the homomorphism $\varphi\colon \mathbb{Q}[x] \to \mathbb{Q} \times \mathbb{Q}$ given by $\varphi(f) = (f(2), f(3))$.

4. Let $R = \mathbb{Z}[x, y]/(x^2 + y^2 - 1)$. Show that homomorphisms $R \to \mathbb{R}$ are naturally in bijection with points of the unit circle

$$\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}.$$

5. What is one question you have about last week's lectures?