Solutions to Homework 5

1. Recall that the quaternions are
\[ \mathbb{H} = \{ x + yi + zj + wk : x, y, z, w \in \mathbb{R} \}, \]
where multiplication is determined by
\[
\begin{align*}
    i^2 &= j^2 = k^2 = -1 \\
    ij &= k = -ji \\
    jk &= i = -kj \\
    ki &= j = -ik.
\end{align*}
\]
Show that the center of \( \mathbb{H} \) is just the real quaternions, i.e. the ones with \( y = z = w = 0 \).

**Solution:** For \( q = x + yi + zj + wk \in \mathbb{H} \), we have
\[
\begin{align*}
    qi &= -y + xi + wj - zk \\
    iq &= -y + xi - wj + zk \\
    qj &= -z - wi + xj + yk \\
    jq &= -z + wi + xj - yk.
\end{align*}
\]
Suppose that \( q \in Z(\mathbb{H}) \). Then \( qi = iq \), so \( w = -w \) and \( -z = z \), so \( w = 0 \) and \( z = 0 \). And \( qj = jq \), so in addition \( y = -y \), so \( y = 0 \).

2. In \( \mathbb{Q}[x] \), let
\[
\begin{align*}
    p &= 2x^7 + 7x^5 - 4x^3 + 9x - 1 \\
    q &= (x - 1)^4.
\end{align*}
\]
Let \( R = \mathbb{Q}[x]/(x^3 - 2x + 1) \).

(a) Find an \( f \in \mathbb{Q}[x] \) of degree \( \leq 2 \) such that \( \bar{p} = \bar{f} \in R \).

**Solution:** We divide \( p \) by \( x^3 - 2x + 1 \) to get a quotient \( 2x^4 + 11x^2 - 2x + 18 \) and a remainder
\[ f = -15x^2 + 47x - 19. \]
(b) Do the same for \( \bar{q}, \bar{p} + \bar{q}, \) and \( \bar{p}\bar{q}. \)

**Solution:** We could just multiply everything out and do polynomial long division, or we could take a few shortcuts as follows.

In \( R \) we have

\[
\bar{x}^3 = 2\bar{x} - 1,
\]

so

\[
(\bar{x} - 1)^3 = \bar{x}^3 - 3\bar{x}^2 + 3\bar{x} - 1 = -3\bar{x}^2 + 5\bar{x} - 2,
\]

so

\[
\bar{q} = (\bar{x} - 1)(\bar{x} - 1)^3 \\
= (\bar{x} - 1)(-3\bar{x}^2 + 5\bar{x} - 2) \\
= -3\bar{x}^3 + 8\bar{x}^2 - 7\bar{x} + 2 \\
= 8\bar{x}^2 - 13\bar{x} + 5.
\]

Next,

\[
\bar{p} + \bar{q} = \bar{p} + \bar{q} = -7\bar{x}^2 + 34\bar{x} - 14.
\]

Finally, for \( \bar{p}\bar{q} = \bar{p}\bar{q} \), we can either multiply the two quadratics and then do long division to get

\[
(-15\bar{x}^2 + 47\bar{x} - 19)(8\bar{x}^2 - 13\bar{x} + 5) \\
= -120\bar{x}^4 + 571\bar{x}^3 - 838\bar{x}^2 + 482\bar{x} - 95, \\
= -1078\bar{x}^2 + 1744\bar{x} - 666,
\]

or we can multiply \( \bar{p} \) by \( \bar{x} - 1 \), reduce mod \( x^3 - 2x + 1 \), multiply by \( \bar{x} - 1 \) again, reduce again, and repeat two more times. It is not clear which is easier.
Clearly this is the kind of thing you’d rather do on a computer. I use a free software package called Macaulay2, available at math.uiuc.edu/Macaulay2. A transcript of my session is below. The i lines are input, and the o lines are output.

\[ \text{i1 : } R = \frac{\mathbb{Q}[x]}{(x^3 - 2x + 1)} \]

\[ o1 = R \]

\[ o1 : \text{QuotientRing} \]

\[ \text{i2 : } p = 2x^7 + 7x^5 - 4x^3 + 9x - 1 \]

\[ o2 = -15x + 47x - 19 \]

\[ o2 : R \]

\[ \text{i3 : } q = (x-1)^4 \]

\[ o3 = 8x - 13x + 5 \]

\[ o3 : R \]

\[ \text{i4 : } p + q \]

\[ o4 = -7x + 34x - 14 \]

\[ o4 : R \]

\[ \text{i5 : } p*q \]

\[ o5 = -1078x + 1744x - 666 \]

\[ o5 : R \]

Sage is another popular free system, available at sagemath.org.
(c) Show that $R$ is not an integral domain. Hint: Factor $x^3 - 2x + 1$.

**Solution:** We observe that $x^3 - 2x + 1 = (x - 1)(x^2 + x - 1)$. Since $x - 1$ and $x^2 + x - 1$ have degree 1 and 2, respectively, they are not multiples of $x^3 - 2x + 1$, so $\bar{x} - 1$ and $\bar{x}^2 + \bar{x} - 1$ are not zero in $R$. But

$$(\bar{x} - 1)(\bar{x}^2 + \bar{x} - 1) = \bar{x}^3 - 2\bar{x} + 1 = 0.$$ 

(d) Show that $\bar{x}$ is a unit in $R$.

**Solution:** Since $\bar{x}^3 - 2\bar{x} + 1 = 0$, we see that

$$1 = -\bar{x}^3 + 2\bar{x} = \bar{x} \cdot (-\bar{x}^2 + 2),$$

so $\bar{x}$ and $-\bar{x}^2 + 2$ are inverse to one another.

3. (a) For $f \in \mathbb{Q}[x]$, show that $f \in (x^2 - 5x + 6)$ if and only if $f(2) = 0$ and $f(3) = 0$.

**Solution:** If $f = (x^2 - 5x + 6) \cdot g$ then $f(2) = 0 \cdot g(2) = 0$ and $f(3) = 0 \cdot g(3) = 0$.

Conversely, suppose that $f(2) = f(3) = 0$. Following the hint, by polynomial long division we can write

$$f = (x^2 - 5x + 6) \cdot q + r$$

with deg $r \leq 1$. Plugging in $x = 2$ and $x = 3$, we see that $r(2) = r(3) = 0$. Write $r = ax + b$; then $2a + b = 0$ and $3a + b = 0$. From this we find that $a = 0$ and $b = 0$, so $r = 0$, so $f = (x^2 - 5x + 6) \cdot q$ as desired.

(b) Show that $\mathbb{Q}[x]/(x^2 - 1) \cong \mathbb{Q} \times \mathbb{Q}$.

**Solution:** This was a typo – I meant to say

$$\mathbb{Q}[x]/(x^2 - 5x + 6) \cong \mathbb{Q} \times \mathbb{Q}.$$ 

Then following the hint, we consider the homomorphism $\varphi : \mathbb{Q}[x] \to \mathbb{Q} \times \mathbb{Q}$ given by $\varphi(f) = (f(2), f(3))$. From part (a) we know that $\ker \varphi = (x^2 - 5x + 6)$, so by the first isomorphism theorem it is enough to show that $\varphi$ is surjective. Given $(a, b) \in \mathbb{Q} \times \mathbb{Q}$, let $f = (b - a)x + (3a - 2b)$. Then $f(2) = a$ and $f(3) = b$, so $\varphi(f) = (a, b)$.

To show that $\mathbb{Q}[x]/(x^2 - 1) \cong \mathbb{Q} \times \mathbb{Q}$ you should take $\varphi(f) = (f(1), f(-1))$.  

4. Let \( R = \mathbb{Z}[x, y]/(x^2 + y^2 - 1) \). Show that homomorphisms \( R \to \mathbb{R} \) are naturally in bijection with points of the unit circle

\[
\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}.
\]

**Solution:** Define a map \( \Phi \) from the set of homomorphisms \( R \to \mathbb{R} \) to the plane \( \mathbb{R}^2 \) by

\[
\Phi(\varphi) = (\varphi(\bar{x}), \varphi(\bar{y})).
\]

First I claim that \( \Phi \) takes values in the circle. For any homomorphism \( \varphi \) we have

\[
\varphi(\bar{x})^2 + \varphi(\bar{y})^2 - 1 = \varphi(x^2 + y^2 - 1) = \varphi(x^2 + y^2 - 1) = \varphi(0) = 0,
\]

so \( (\varphi(\bar{x}), \varphi(\bar{y})) \) is in the circle.

Next I claim that \( \Phi \) is injective. If two homomorphisms \( \varphi, \psi \) satisfy \( \varphi(\bar{x}) = \psi(\bar{x}) \) and \( \varphi(\bar{y}) = \psi(\bar{y}) \), then for all

\[
f = \sum a_{ij} x^i y^j \in \mathbb{Z}[x, y]
\]

we have

\[
\varphi(\bar{f}) = \sum a_{ij} \varphi(\bar{x})^i \varphi(\bar{y})^j = \sum a_{ij} \psi(\bar{x})^i \psi(\bar{y})^j = \psi(\bar{f}),
\]

so \( \varphi = \psi \).

Finally I claim that \( \Phi \) surjects onto the circle. Given \( (a, b) \in \mathbb{R}^2 \) with \( a^2 + b^2 = 1 \), define a map \( \varphi : R \to \mathbb{R} \) by \( \varphi(\bar{f}) = f(a, b) \). This is well-defined, because if \( \bar{g} = \bar{f} \) then \( g - f = (x^2 + y^2 - 1) \cdot h \) for some \( h \in \mathbb{Z}[x, y] \), so

\[
g(a, b) - f(a, b) = (a^2 + b^2 - 1) \cdot h(a, b) = 0,
\]

so \( g(a, b) = f(a, b) \). It is straightforward to check that \( \varphi \) is a homomorphism. And clearly \( \varphi(\bar{x}) = a \) and \( \varphi(\bar{y}) = b \), so \( \Phi(\varphi) = (a, b) \).

Notice that there was nothing special about \( \mathbb{R} \) here: for any ring \( S \), the set of homomorphisms \( R \to S \) is in bijection with the set of pairs \( (a, b) \in S \times S \) with \( a^2 + b^2 = 1 \).