Solutions to Homework 5

1. Recall that the *quaternions* are

$$\mathbb{H} = \{x + yi + zj + wk : x, y, z, w \in \mathbb{R}\},$$

where multiplication is determined by

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ ij = k &= -ji \\ jk = i &= -kj \\ ki = j &= -ik. \end{aligned}$$

Show that the center of $\mathbb{H}$ is just the *real* quaternions, i.e. the ones with with $y = z = w = 0$.

Solution: For $q = x + yi + zj + wk \in \mathbb{H}$, we have

$$\begin{aligned} qi &= -y + xi + wj - zk \\ iq &= -y + xi - wj + zk \end{aligned}$$

$$\begin{aligned} qj &= -z - wi + xj + yk \\ jq &= -z + wi + xj - yk. \end{aligned}$$

Suppose that $q \in Z(\mathbb{H})$. Then $qi = iq$, so $w = -w$ and $-z = z$, so $w = 0$ and $z = 0$. And $qj = jq$, so in addition $y = -y$, so $y = 0$.

2. In $\mathbb{Q}[x]$, let

$$\begin{aligned} p &= 2x^7 + 7x^5 - 4x^3 + 9x - 1 \\ q &= (x - 1)^4. \end{aligned}$$

Let $R = \mathbb{Q}[x]/(x^3 - 2x + 1)$.

(a) Find an $f \in \mathbb{Q}[x]$ of degree ≤ 2 such that $\bar{p} = \bar{f} \in R$.
Solution: We divide p by $x^3 - 2x + 1$ to get a quotient $2x^4 + 11x^2 - 2x + 18$ and a remainder

$$f = -15x^2 + 47x - 19.$$

(b) Do the same for $\bar{q}$, $\overline{p+q}$, and $\overline{pq}$.

Solution: We could just multiply everything out and do polynomial long division, or we could take a few shortcuts as follows. In R we have

$$\bar{x}^3 = 2\bar{x} - 1,$$

so

$$\begin{aligned}(\bar{x}-1)^3 &= \bar{x}^3 - 3\bar{x}^2 + 3\bar{x} - 1 \\ &= -3\bar{x}^2 + 5\bar{x} - 2,\end{aligned}$$

so

$$\begin{aligned}\bar{q} &= (\bar{x}-1)(\bar{x}-1)^3 \\ &= (\bar{x}-1)(-3\bar{x}^2 + 5\bar{x} - 2) \\ &= -3\bar{x}^3 + 8\bar{x}^2 - 7\bar{x} + 2 \\ &= 8\bar{x}^2 - 13\bar{x} + 5.\end{aligned}$$

Next,

$$\overline{p+q} = \bar{p} + \bar{q} = -7\bar{x}^2 + 34\bar{x} - 14.$$

Finally, for $\overline{pq} = \bar{p}\bar{q}$, we can either multiply the two quadratics and then do long division to get

$$\begin{aligned}&(-15\bar{x}^2 + 47\bar{x} - 19)(8\bar{x}^2 - 13\bar{x} + 5) \\ &= -120\bar{x}^4 + 571\bar{x}^3 - 838\bar{x}^2 + 482\bar{x} - 95, \\ &= -1078\bar{x}^2 + 1744\bar{x} - 666,\end{aligned}$$

or we can multiply $\bar{p}$ by $\bar{x} - 1$, reduce mod $x^3 - 2x + 1$, multiply by $\bar{x} - 1$ again, reduce again, and repeat two more times. It is not clear which is easier.

Clearly this is the kind of thing you'd rather do on a computer. I use a free software package called Macaulay2, available at `math.uiuc.edu/Macaulay2`. A transcript of my session is below. The `i` lines are input, and the `o` lines are output.

```
i1 : R = QQ[x]/(x^3 - 2*x + 1)

o1 = R

o1 : QuotientRing

i2 : p = 2*x^7 + 7*x^5 - 4*x^3 + 9*x - 1

          2
o2 = - 15x  + 47x - 19

o2 : R

i3 : q = (x-1)^4

       2
o3 = 8x  - 13x + 5

o3 : R

i4 : p + q

         2
o4 = - 7x  + 34x - 14

o4 : R

i5 : p*q

            2
o5 = - 1078x  + 1744x - 666

o5 : R
```

Sage is another popular free system, available at `sagemath.org`.

(c) Show that R is not an integral domain. Hint: Factor $x^3 - 2x + 1$.
Solution: We observe that $x^3 - 2x + 1 = (x - 1)(x^2 + x - 1)$. Since $x - 1$ and $x^2 + x - 1$ have degree 1 and 2, respectively, they are not multiples of $x^3 - 2x + 1$, so $\bar{x} - 1$ and $\bar{x}^2 + \bar{x} - 1$ are not zero in R. But

$$(\bar{x} - 1)(\bar{x}^2 + \bar{x} - 1) = \bar{x}^3 - 2\bar{x} + 1 = 0.$$

(d) Show that $\bar{x}$ is a unit in R.
Solution: Since $\bar{x}^3 - 2\bar{x} + 1 = 0$, we see that

$$1 = -\bar{x}^3 + 2\bar{x} = \bar{x} \cdot (-\bar{x}^2 + 2),$$

so $\bar{x}$ and $-\bar{x}^2 + 2$ are inverse to one another.

3. (a) For $f \in \mathbb{Q}[x]$, show that $f \in (x^2 - 5x + 6)$ if and only if $f(2) = 0$ and $f(3) = 0$.
Solution: If $f = (x^2 - 5x + 6) \cdot g$ then $f(2) = 0 \cdot g(2) = 0$ and $f(3) = 0 \cdot g(3) = 0$.
Conversely, suppose that $f(2) = f(3) = 0$. Following the hint, by polynomial long division we can write

$$f = (x^2 - 5x + 6) \cdot q + r$$

with $\deg r \leq 1$. Plugging in $x = 2$ and $x = 3$, we see that $r(2) = r(3) = 0$. Write $r = ax + b$; then $2a + b = 0$ and $3a + b = 0$. From this we find that $a = 0$ and $b = 0$, so $r = 0$, so $f = (x^2 - 5x + 6) \cdot q$ as desired.

(b) Show that $\mathbb{Q}[x]/(x^2 - 1) \cong \mathbb{Q} \times \mathbb{Q}$.
Solution: This was a typo – I meant to say

$$\mathbb{Q}[x]/(x^2 - 5x + 6) \cong \mathbb{Q} \times \mathbb{Q}.$$

Then following the hint, we consider the homomorphism $\varphi\colon \mathbb{Q}[x] \to \mathbb{Q} \times \mathbb{Q}$ given by $\varphi(f) = (f(2), f(3))$. From part (a) we know that $\ker \varphi = (x^2 - 5x + 6)$, so by the first isomorphism theorem it is enough to show that φ is surjective. Given $(a, b) \in \mathbb{Q} \times \mathbb{Q}$, let $f = (b - a)x + (3a - 2b)$. Then $f(2) = a$ and $f(3) = b$, so $\varphi(f) = (a, b)$.
To show that $\mathbb{Q}[x]/(x^2 - 1) \cong \mathbb{Q} \times \mathbb{Q}$ you should take $\varphi(f) = (f(1), f(-1))$.

4. Let $R = \mathbb{Z}[x,y]/(x^2+y^2-1)$. Show that homomorphisms $R \to \mathbb{R}$ are naturally in bijection with points of the unit circle

$$\{(a,b) \in \mathbb{R}^2 : a^2+b^2=1\}.$$

Solution: Define a map Φ from the set of homomorphisms $R \to \mathbb{R}$ to the plane $\mathbb{R}^2$ by

$$\Phi(\varphi) = (\varphi(\bar{x}), \varphi(\bar{y})).$$

First I claim that Φ takes values in the circle. For any homomorphism φ we have

$$\varphi(\bar{x})^2 + \varphi(\bar{y})^2 - 1 = \varphi(\bar{x}^2+\bar{y}^2-1) = \varphi(\overline{x^2+y^2-1}) = \varphi(0) = 0,$$

so $(\varphi(\bar{x}), \varphi(\bar{y}))$ is in the circle.

Next I claim that Φ is injective. If two homomorphisms φ, ψ satisfy $\varphi(\bar{x}) = \psi(\bar{x})$ and $\varphi(\bar{y}) = \psi(\bar{y})$, then for all

$$f = \sum a_{ij} x^i y^j \in \mathbb{Z}[x,y]$$

we have

$$\varphi(\bar{f}) = \sum a_{ij} \varphi(\bar{x})^i \varphi(\bar{y})^j = \sum a_{ij} \psi(\bar{x})^i \psi(\bar{y})^j = \psi(\bar{f}),$$

so $\varphi = \psi$.

Finally I claim that Φ surjects onto the circle. Given $(a,b) \in \mathbb{R}^2$ with $a^2+b^2=1$, define a map $\varphi\colon R \to \mathbb{R}$ by $\varphi(\bar{f}) = f(a,b)$. This is well-defined, because if $\bar{g} = \bar{f}$ then $g - f = (x^2+y^2-1)\cdot h$ for some $h \in \mathbb{Z}[x,y]$, so

$$g(a,b) - f(a,b) = (a^2+b^2-1)\cdot h(a,b) = 0,$$

so $g(a,b) = f(a,b)$. It is straightforward to check that φ is a homomorphism. And clearly $\varphi(\bar{x}) = a$ and $\varphi(\bar{y}) = b$, so $\Phi(\varphi) = (a,b)$.

Notice that there was nothing special about $\mathbb{R}$ here: for any ring S, the set of homomorphisms $R \to S$ is in bijection with the set of pairs $(a,b) \in S \times S$ with $a^2+b^2=1$.