# Solutions to Homework 6

1. Let $R$ be a commutative ring and let $I, J \subset R$ be ideals. Earlier you showed that $I+J$, $IJ$, and $I \cap J$ are ideals, and that $IJ \subset I \cap J$.

   (a) Show that if $I + J = (1)$ then $IJ = I \cap J$.
   **Solution:** We know that $IJ \subset I \cap J$, so it remains to show that $I \cap J \subset IJ$. Since $I + J = (1)$, there are elements $i \in I$ and $j \in J$ such that $i + j = 1$. Let $k \in I \cap J$, and multiply $i + j = 1$ through by $k$ to get $ki + kj = k$. Write this more suggestively as

   $$k = ik + kj.$$

   The first term is in $IJ$ because $k \in J$, and the second term is in $IJ$ because $k \in I$, so $k \in IJ$ as desired.

   (b) Show that the converse of part (a) does not hold in general by taking $R = \mathbb{Z}[x]$, $I = (2)$, and $J = (x)$.
   **Solution:** First I claim that $IJ = I \cap J$. We know that

   $$2x \in IJ \subset I \cap J,$$

   so it is enough to show that $I \cap J \subset (2x)$ as well. Let $f \in I$, so we can write

   $$f = 2a_0 + 2a_1 x + 2a_2 x^2 + \cdots + 2a_n x^n.$$

   for some $a_0, \ldots, a_n \in \mathbb{Z}$. If $f \in J$ as well then $x \mid f$, so $a_0 = 0$; so we have

   $$f = 2x \cdot (a_1 + a_2 x + \cdots + a_n x^{n-1}) \in (2x),$$

   as desired.
   Next I claim that $I + J = (2, x)$ is different from $(1)$, and in fact that $1 \notin I + J$. Suppose we could write $1 = 2f + xg$ for some $f, g \in \mathbb{Z}[x]$. Plugging in $x = 0$, we get $1 = 2f(0) + 0g(0)$, but the left-hand side is odd while the right-hand side is even.

2. Let $R$ be a commutative ring, let $a, b, c, d \in R$, and consider the ideals $I = (a, b)$ and $J = (c, d)$.

   (a) Show that $I + J = (a, b, c, d)$.

   **Solution:** By definition, $I$ is the set of linear combinations $ra + sb$, where $r, s \in R$, and $J$ is the set of linear combinations $tc + ud$, where $t, u \in R$. Thus $I + J$ is the set of all $ra + sb + tc + ud$, which is exactly $(a, b, c, d)$.

   (b) Show that $IJ = (ac, ad, bc, bd)$.

   **Solution:** Clearly $ac$, $ad$, $bc$, and $bd$ are in $IJ$, so $(ac, ad, bc, bd) \subset IJ$. Conversely, to show that $IJ \subset (ac, ad, bc, bd)$ it is enough to show that for every $i \in I$ and $j \in J$ we have $ij \in (ac, ad, bc, bd)$, since $IJ$ is the set of sums terms of the form $ij$ and the ideal $(ac, ad, bc, bd)$ is closed under taking sums. So let $i = ra + sb \in I$ and $j = tc + ud \in J$; then

$$ij = (ra + sb)(tc + ud) = rt \cdot ac + ru \cdot ad + st \cdot bc + su \cdot bd,$$

   which is indeed in $(ac, ad, bc, bd)$.

3. Let $R = \mathbb{Z}[\sqrt{-5}]$, and consider the ideals

$$\begin{aligned} I_1 &= (2, 1 + \sqrt{-5}) \\ I_2 &= (2, 1 - \sqrt{-5}) \\ I_3 &= (1 + \sqrt{-5}, 1 - \sqrt{-5}) \\ J_1 &= (3, 1 + \sqrt{-5}) \\ J_2 &= (3, 1 - \sqrt{-5}). \end{aligned}$$

   (a) Show that $I_1 = I_2 = I_3$.

   **Solution:** It is enough to show that $1-\sqrt{-5} \in I_1$, that $1+\sqrt{-5} \in I_2$, and that $2 \in I_3$, for then each ideal contains the generators of the other two. We verify this:

$$\begin{aligned} 1 - \sqrt{-5} &= (1 + \sqrt{-5}) + 2 \cdot (-\sqrt{-5}) \in I_1 \\ 1 + \sqrt{-5} &= (1 - \sqrt{-5}) + 2 \cdot \sqrt{-5} \in I_2 \\ 2 &= (1 + \sqrt{-5}) + (1 - \sqrt{-5}) \in I_3. \end{aligned}$$

(b) Show that $I_1 + J_1$, $I_1 + J_2$, and $J_1 + J_2$ are all equal to (1).

**Solution:** Clearly all three are contained in (1), which is the whole ring, so it is enough to show that all three contain 1. For $I_1 + J_1$ and $I_1 + J_2$, we have $1 = -2 + 3$. For $J_1 + J_2$, observe that $2 = (1 + \sqrt{-5}) + (1 - \sqrt{-5}) \in J_1 + J_2$, and $3 \in J_1 + J_2$, so again $1 = 3 - 2$ is in $J_1 + J_2$.

(c) Find $J_1J_2$, $I_1J_1$, $I_2J_2$, and $I_1I_2$. Hint: They are all principal ideals, i.e. they can be generated by one element.

**Solution:** By problem 2b we have

$$J_1J_2 = (9, 3 + 3\sqrt{-5}, 3 - 3\sqrt{-5}, 6).$$

This contains $9 - 6 = 3$, so $(3) \subset J_1J_2$. But every generator of $J_1J_2$ is a multiple of 3, so $J_1J_2 \subset (3)$ as well. Thus $J_1J_2 = (3)$. Next we have

$$I_1J_1 = (6, 2 + 2\sqrt{-5}, 3 + 3\sqrt{-5}, (1 + \sqrt{-5})^2).$$

This contains $1+\sqrt{-5} = (3+3\sqrt{-5})-(2+2\sqrt{-5})$, and conversely every generator of $I_1J_1$ is a multiple of $1 + \sqrt{-5}$, so

$$I_1J_1 = (1 + \sqrt{-5}).$$

Similarly we find that

$$I_2J_2 = (1 - \sqrt{-5}).$$

Finally,

$$I_1I_2 = (4, 2 + 2\sqrt{-5}, 2 - 2\sqrt{-5}, 6).$$

This contains $6 - 4 = 2$, and conversely every generator of $I_1J_1$ is a multiple of 2, so

$$I_1I_2 = (2).$$

4. Let $I_1, J_1, J_2 \subset R$ be as in the previous problem.

(a) Show that $R/I_1 \cong \mathbb{Z}/2$. Hint: In lecture, in week 4 or so when we first discussed ideals, we wrote down a homomorphism $\varphi\colon R \to \mathbb{Z}/2$ with $\ker\varphi = I_1$.

**Solution:** The map was $\varphi(a + b\sqrt{-5}) = a + b \pmod 2$. It is straightforward to check that this is a homomorphism, because $1^2 \equiv -5 \pmod 2$. We have $\varphi(2) = 0$ and $\varphi(1 + \sqrt{-5}) = 0$, so

$I_1 \subset \ker\varphi$. Conversely, let $a + b\sqrt{-5} \in \ker\varphi$; then $a + b \equiv 0 \pmod 2$, so we can write $a + b = 2c$ for some $c \in \mathbb{Z}$, so

$$a + b\sqrt{-5} = a - b + b + b\sqrt{-5} = 2(c - b) + b(1 + \sqrt{-5}) \in I_1.$$

The map $\varphi$ is surjective, so by the first isomorphism theorem we have $R/I_1 \cong \mathbb{Z}/2$.

(b) Show that $R/J_1 \cong \mathbb{Z}/3$, and $R/J_2 \cong \mathbb{Z}/3$.

**Solution:** Again it is enough to give a surjective homomorphism $\varphi\colon R \to \mathbb{Z}/3$ with $\ker\varphi = J_1$ or $J_2$. In homework 3, problem 4, we saw that giving a homomorphism $\mathbb{Z}[i] \to \mathbb{Z}/5$ was the same as giving an element of $d \in \mathbb{Z}/5$ with $d^2 = -1$. Similarly, giving a homomorphism $\mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}/3$ is the same as giving an element $d \in \mathbb{Z}/3$ with $d^2 = -5$. In $\mathbb{Z}/3$ we have $-5 = 1$, so we can take $d = 1$ or $d = 2$, which gives two homomorphisms:

$$\varphi_1(a + b\sqrt{-5}) = a + b \pmod 3$$
$$\varphi_2(a + b\sqrt{-5}) = a + 2b \pmod 3.$$

Let us show that $\ker\varphi_1 = J_2$. We have $\varphi_1(3) = 0$ and $\varphi_1(1 - \sqrt{-5}) = 0$, so $J_2 \subset \ker\varphi_1$. Conversely, let $a + b\sqrt{-5} \in \ker\varphi_1$; then $a + b \equiv 0 \pmod 3$, so we can write $a + b = 3c$ for some $c \in \mathbb{Z}$, so

$$a + b\sqrt{-5} = a + b - b + b\sqrt{-5} = 3c - b(1 - \sqrt{-5}) \in J_2.$$

The proof that $\ker\varphi_2 = J_1$ is entirely similar, especially if you write

$$\varphi_2(a + b\sqrt{-5}) = a - b \pmod 3.$$

(c) Show that $R/(3) \cong \mathbb{Z}/3 \times \mathbb{Z}/3$.

Hint: Use the Chinese Remainder Theorem.

**Solution:** In problem 5b we saw that $J_1 + J_2 = (1)$; thus by the Chinese Remainder Theorem we have

$$R/(J_1 \cap J_2) \cong R/J_1 \times R/J_2.$$

By problems 1a, 3b, and 3c, we have $J_1 \cap J_2 = J_1J_2 = (3)$. By problem 4b, the right-hand side is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$.