Homework 9

Not due, but any of these might appear on the final.

1. Show that $\mathbb{Z}_2[x]/(x^3 + x + 1) \cong \mathbb{Z}_2[y]/(y^3 + y^2 + 1)$.

2. List all the irreducible polynomials of degree 4 in $\mathbb{Z}_2[x]$.

   Hint: First rule out the ones with $f(0) = 0$ or $f(1) = 0$, but don't forget that f might factor as a product of two irreducible quadratics.

3. Show that $f = x^4 + 3x^3 + 5x^2 + 7x + 9$ is irreducible in $\mathbb{Q}[x]$.

   Hint: First argue that f were reducible in $\mathbb{Z}[x]$ then $\bar{f} \in \mathbb{Z}_2[x]$ would be reducible. Then appeal to Gauss's lemma.

4. Let $f \in \mathbb{Z}[x]$, and suppose that $f(\frac{1}{2}) = 0$. Show that $2x - 1 \mid f$.

   Hint: Consider using Gauss's lemma.

5. We have seen that $R = \mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain, much less a principal ideal domain. Show nonetheless that every non-zero prime ideal in R is maximal.

   Hint: One possibility is to argue that for every non-zero ideal $I \subset R$, the quotient R/I is a finite ring; then from homework 2 we know that a finite integral domain is a field. But I don't promise that this is the best proof.

6. We define the *field of formal Laurent series* $\mathbb{Q}((x))$, which is like the ring of formal power series $\mathbb{Q}[[x]]$ but we allow finitely many negative exponents:

   $$\mathbb{Q}((x)) = \{a_{-n}x^{-n} + \cdots + a_{-1}x^{-1} + a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\}.$$

   Let's take for granted that $\mathbb{Q}((x))$ is a ring. Show that every non-zero element has an inverse, so $\mathbb{Q}((x))$ is a field. Show that the field of fractions of $\mathbb{Q}[[x]]$ is isomorphic to $\mathbb{Q}((x))$.