Midterm 1

Friday, October 21, 2016

1. The *center* of a ring R is the subset

$$Z = \{z \in R : zx = xz \text{ for all } x \in R\}.$$

   Show that Z is a subring of R.

2. (a) Show that if R is an integral domain and $x^2 = 1$, then $x = 1$ or $x = -1$.

   (b) Find all $x \in \mathbb{Z}/8$ with $x^2 = 1$.

3. Show that $\mathbb{Z}/4$ is not isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

4. Let R be the ring of all functions on $[0, 1]$, and $S \subset R$ the subring of continuous functions. Show that the function $f(x) = x^2$ is a zero-divisor in R, but not in S. Show that it is not a unit in either ring.

5. An element x in a ring R is called *nilpotent* if there is a positive integer n such that $x^n = 0$.

   (a) For example, show that the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{R})$$

   is nilpotent.

   (b) Show that if x is nilpotent then $1 - x$ is a unit. Hint: To find the inverse, start writing the power series for $1/(1 - x)$, and observe that you can stop at some point.

   (c) Now suppose that R is commutative. Show that if x is nilpotent and u is a unit then $u + x$ is a unit.