Solutions to Midterm 1

1. The *center* of a ring R is the subset
$$Z = \{z \in R : zx = xz \text{ for all } x \in R\}.$$
Show that Z is a subring of R.

Solution:

First we show that $1 \in Z$: for all $x \in R$ we have $1 \cdot x = x = x \cdot 1$.

Next we show that if $z, w \in Z$ then $z + w \in Z$: for all $x \in R$ we have
$$(z + w)x = zx + wx = xz + wz = x(w + z).$$
Next we show that if $z, w \in Z$ then $zw \in Z$: for all $x \in R$ we have
$$zwx = zxw = xzw.$$
Last we show that if $z \in Z$ then $-z \in Z$. For all $x \in R$ we have
$$(-z)x = -(zx) = -(xz) = x(-z),$$
where we have used some facts from the first homework.

2. (a) Show that if R is an integral domain and $x^2 = 1$, then $x = 1$ or $x = -1$.
Solution: We have
$$(x - 1)(x + 1) = x^2 - 1 = 0.$$
Since R is an integral domain, we conclude that $x - 1 = 0$ or $x + 1 = 0$. In the first case we have $x = 1$, and in the second case we have $x = -1$.

(b) Find all $x \in \mathbb{Z}/8$ with $x^2 = 1$.
Solution: In $\mathbb{Z}/8$ we have
$$1^2 = 1 \qquad 3^2 = 9 = 1 \qquad 5^2 = 25 = 1 \qquad 7^2 = 49 = 1.$$

3. Show that $\mathbb{Z}/4$ is not isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

   Solution: Let $\varphi\colon \mathbb{Z}/4 \to \mathbb{Z}/2 \times \mathbb{Z}/2$ be a homomorphism. Then $\varphi(1) = (1, 1)$, so

   $$\begin{aligned}\varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) \\ &= (1,1) + (1,1) = (1+1, 1+1) = (0,0) = \varphi(0),\end{aligned}$$

   so φ is not injective. In particular there is no bijective homomorphism from $\mathbb{Z}/4$ to $\mathbb{Z}/2 \times \mathbb{Z}/2$, i.e. no isomorphism.

   Alternatively, you could show that there is no homomorphism at all from $\mathbb{Z}/2 \times \mathbb{Z}/2$ to $\mathbb{Z}/4$.

   Later in the term we will classify (up to isomorphism) all commutative rings with four elements.

4. Let R be the ring of all functions on $[0, 1]$, and $S \subset R$ the subring of continuous functions. Show that the function $f(x) = x^2$ is a zero-divisor in R, but not in S. Show that it is not a unit in either ring.

   Solution: First we argue that f is a zero-divisor in R. Let

   $$g(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Then $f \cdot g = 0$.

   Next we argue that f is not a zero-divisor in S. Let $g\colon [0, 1] \to \mathbb{R}$ be continuous, and suppose that $f(x)g(x) = 0$ for all $x \in [0, 1]$. For $x \neq 0$ we have $f(x) \neq 0$, so $g(x) = 0$. But g is continuous, so $g(0) = 0$ as well.

   Next we argue that f is not a unit in either ring. Let $g\colon [0, 1] \to \mathbb{R}$. Then $f(0) \cdot g(0) = 0 \cdot g(0) = 0$, so $f \cdot g$ is not the constant function 1.

5. An element x in a ring R is called *nilpotent* if there is a positive integer n such that $x^n = 0$.

   (a) For example, show that the matrix

   $$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{R})$$

   is nilpotent.

   Solution: We find that

   $$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

   and

   $$A^3 = A \cdot A^2 = 0.$$

   (b) Show that if x is nilpotent then $1 - x$ is a unit. Hint: To find the inverse, start writing the power series for $1/(1-x)$, and observe that you can stop at some point.

   Solution: Let

   $$y = 1 + x + x^2 + \cdots + x^{n-1}.$$

   Then we have

   $$\begin{aligned} (1-x) \cdot y &= 1 + x + x^2 + \cdots + x^{n-1} - x - x^2 - x^3 - \cdots - x^n \\ &= 1 - x^n \\ &= 1 - 0 \\ &= 1, \end{aligned}$$

   and similarly $y \cdot (1-x) = 1$.

   (c) Now suppose that R is commutative. Show that if x is nilpotent and u is a unit then $u + x$ is a unit.

   Solution: Let

   $$y = u^{-1} - u^{-2}x + u^{-3}x^2 - \cdots + (-1)^{n-1}u^{-n}x^{n-1}.$$

   Then we have

   $$\begin{aligned} (u+x) \cdot y = 1 &- u^{-1}x + u^{-2}x^2 - \cdots + (-1)^{n-1}u^{-n+1}x^{n-1} \\ &+ u^{-1}x - u^{-2}x^2 + u^{-3}x^3 - \cdots + (-1)^{n-1}u^{-n}x^n \\ = 1 &+ (-1)^{n-1}x^n = 1 + 0 = 1. \end{aligned}$$