Midterm 2

Friday, November 11, 2016

1. Show that $\mathbb{Q}[x]/(x-22) \cong \mathbb{Q}$.
2. Recall that an element x in a ring R is called *nilpotent* if there is a positive integer n such that $x^n = 0$. Let $N \subset R$ be the set of nilpotent elements.
   (a) Show that if R is commutative then N is an ideal.
   (b) Suppose N is an ideal, so we can form the quotient ring R/N. Show that 0 is the only nilpotent element of R/N.
   (c) Show that if R is non-commutative then N need not be an ideal. Hint: In $M_2(\mathbb{R})$, consider the elements
   $$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
   and their sum.
3. An element e in a ring R is called *idempotent* if $e^2 = e$.
   (a) List the idempotents in $\mathbb{Z}/12$.
   (b) Where do they map to under the isomorphism $\mathbb{Z}/12 \cong \mathbb{Z}/3 \times \mathbb{Z}/4$?
   (c) Show that if $e \in R$ is idempotent then $1-e$ is idempotent.
   (d) If R is commutative and $e \in R$ is idempotent, use the Chinese Remainder Theorem to show that
   $$R \cong R/(e) \times R/(1-e).$$
   Hint: On the last homework you showed that if $I + J = (1)$ then $I \cap J = IJ$.
4. Let $S = \mathbb{Z}[2i]$, which is a subring of $\mathbb{Z}[i]$.
   (a) Show that there is no $z \in S$ with $|z|^2 = 2$.
   (b) Show that 2 is irreducible in S.
   (c) Show that 2 is not prime in S. Hint: $4 = 2 \cdot 2 = (2i)(-2i)$.
   (d) Show that the ideal $(2, 2i) \subset S$ is not principal.
   (e) Show that $S/(2, 2i) \cong \mathbb{Z}/2$.
   Hint: Consider $\varphi(a + 2bi) = a \pmod 2$.