Solutions to Midterm 2

1. Show that $\mathbb{Q}[x]/(x-22) \cong \mathbb{Q}$.

   Solution: Consider the homomorphism $\varphi\colon \mathbb{Q}[x] \to \mathbb{Q}$ given by

   $$\varphi(f) = f(22).$$

   The first isomorphism theorem states that $\mathbb{Q}[x]/\ker\varphi \cong \operatorname{im}\varphi$. I claim that $\ker\varphi = (x-22)$ and $\operatorname{im}\varphi = \mathbb{Q}$.

   First, $\ker\varphi = (x-22)$. For any $f \in \mathbb{Q}[x]$ we can write

   $$f = (x-22)\cdot q + r,$$

   where $q, r \in \mathbb{Q}[x]$ and $\deg r < 1$, that is, r is a constant. Plugging in $x = 22$, we find that $r = f(22)$. Thus $f(22) = 0$ if and only if $f = (x-22)\cdot q$ for some $q \in \mathbb{Q}[x]$: that is, $f \in \ker\varphi$ if and only if $f \in (x-22)$.

   Second, $\operatorname{im}\varphi = \mathbb{Q}$. Indeed, for all $a \in \mathbb{Q}$ we have $\varphi(x-22+a) = a$.

2. Recall that an element x in a ring R is called *nilpotent* if there is a positive integer n such that $x^n = 0$. Let $N \subset R$ be the set of nilpotent elements.

   (a) Show that if R is commutative then N is an ideal.

   Solution: Let $x, y \in N$, and suppose that $x^n = 0$ and $y^m = 0$ for some positive integers n and m. Since R is commutative, when we multiply out $(x+y)^{n+m}$ we get a linear combination

   $$x^{n+m},\ x^{n+m-1}y,\ x^{n+m-2}y^2,\ \ldots,\ x^n y^m,\ x^{n-1}y^{m+1},\ \ldots,\ y^{m+n}.$$

   The first $m+1$ terms vanish because $x^n = 0$. The remaining terms vanish because $y^m = 0$. Thus $x + y \in N$.

   Next let $x \in N$ with $x^n = 0$, and let $r \in R$ be arbitrary. Since R is commutative, we have

   $$(rx)^n = r^n x^n = 0,$$

   so $rx \in N$.

   (b) Suppose N is an ideal, so we can form the quotient ring R/N.

   Show that 0 is the only nilpotent element of R/N.

   Solution: Let $x \in R$, and suppose that $\bar{x}^n = 0 \in R/N$ for some positive integer n. Then $x^n \in N$, so there is a positive integer m such that $(x^n)^m = 0$, so $x^{nm} = 0$, so $x \in N$, so $\bar{x} = 0$.

(c) Show that if R is non-commutative then N need not be an ideal. Hint: In $M_2(\mathbb{R})$, consider the elements

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and their sum.

Solution: We have $A^2 = 0 = B^2$, so $A \in N$ and $B \in N$. But

$$A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

satisfies $(A+B)^2 = 1$. Thus $(A+B)^n$ equals $A+B$ if n is odd, or 1 if n is even, and in particular is never 0. Thus $A+B \notin N$, so N is not an ideal.

Indeed, we have seen that the only ideals in $M_2(\mathbb{R})$ are (0) and (1).

3. An element e in a ring R is called *idempotent* if $e^2 = e$.

(a) List the idempotents in $\mathbb{Z}/12$.

Solution: 0, 1, 4, 9.

(b) Where do they map to under the isomorphism $\mathbb{Z}/12 \cong \mathbb{Z}/3 \times \mathbb{Z}/4$?

Solution: $(0,0)$, $(1,1)$, $(1,0)$, $(0,1)$.

(c) Show that if $e \in R$ is idempotent then $1 - e$ is idempotent.

Solution: $(1-e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$.

(d) If R is commutative and $e \in R$ is idempotent, use the Chinese Remainder Theorem to show that

$$R \cong R/(e) \times R/(1-e).$$

Hint: On the last homework you showed that if $I + J = (1)$ then $I \cap J = IJ$.

Solution: Let $I = (e)$ and $J = (1-e)$. Then $1 = e + 1 - e \in I + J$, so $I + J = (1)$. Thus the Chinese Remainder Theorem gives

$$R/(I \cap J) \cong R/I \times R/J.$$

Since R is commutative, from the last homework we have $I \cap J = IJ$. From the same homework we have

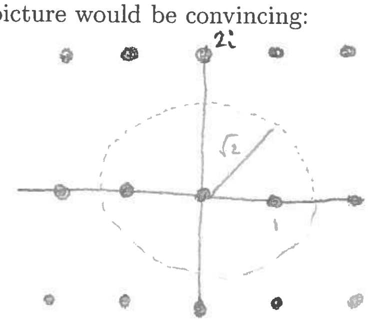
$$IJ = (e \cdot (1-e)) = (e - e^2) = (0),$$

so $R/(I \cap J) = R/(0) \cong R$.

4. Let $S = \mathbb{Z}[2i]$, which is a subring of $\mathbb{Z}[i]$.

(a) Show that there is no $z \in S$ with $|z|^2 = 2$.

Solution: A picture would be convincing:



If you prefer an algebraic argument, let $z = a+2bi$, so $|z|^2 = a^2+4b^2$. If $b \neq 0$ then $|z|^2 \geq 4$, so $|z|^2 \neq 2$. If $b = 0$ then $|a|$ can be 0, 1, or ≥ 2, so $|z|^2 = a^2$ can be 0, 1, or ≥ 4, and so again $|z|^2 \neq 2$.

(b) Show that 2 is irreducible in S.

Solution: For any $z \in S$, observe that $|z|^2$ is a non-negative integer, and that if $|z|^2 = 1$ then $z = \pm 1$, so z is a unit.

Now suppose that $2 = zw$ with $z, w \in S$. Then $4 = |z|^2|w|^2$. If $|z|^2 = 1$ and $|w|^2 = 4$ then z is a unit. If $|z|^2 = 4$ and $|w|^2 = 1$ then w is a unit. And we have seen that $|z|^2 = 2$, $|w|^2 = 2$ is impossible.

(c) Show that 2 is not prime in S. Hint: $4 = 2 \cdot 2 = (2i)(-2i)$.

Solution: Following the hint, we observe that $2 \mid (2i)(-2i)$. If 2 were prime then we would have $2 \mid 2i$ or $2 \mid -2i$. But $\pm 2i$ is not a multple of 2: indeed, for any $z = a + 2bi \in S$ we have $2z = 2a + 4bi$, which cannot equal $\pm 2i$ because ± 2 is not a multiple of 4.

(d) Show that the ideal $(2, 2i) \subset S$ is not principal.

Solution: Suppose on the contrary that $(2, 2i) = (z)$ for some $z \in S$. Then $z \mid 2$, so $2 = zw$ for some $w \in S$, so either z is a unit or w is a unit. If z is a unit then $(2, 2i) = (1)$; but we see that $1 \notin (2, 2i)$. If w is a unit then $(z) = (2)$, so in particular $2i \in (2)$, but we have just seen that $2 \nmid 2i$.

(e) Show that $S/(2, 2i) \cong \mathbb{Z}/2$.

Hint: Consider $\varphi(a + 2bi) = a \pmod 2$.

Solution: Following the hint, we consider the map $\varphi \colon S \to \mathbb{Z}/2$ defined by $\varphi(a + 2bi) = a \pmod 2$. It is straightforward to check that φ is a homomorphism. It is also clear that φ is surjective: $\varphi(0) = 0$ and $\varphi(1) = 1$. So it remains to show that $\ker \varphi = (2, 2i)$. We have $\varphi(2) = 0$ and $\varphi(2i) = 0$, so $(2, 2i) \subset \ker \varphi$. Conversely, if $\varphi(a + 2bi) = 0$ then $a \equiv 0 \pmod 2$, so we can write $a = 2c$ for some $c \in \mathbb{Z}$, so $a + 2bi = 2c + (2i)b \in (2, 2i)$.