Homework 1

Due Wednesday, January 18, 2017

Math 545 students must do all the "optional" parts.

1. (a) Show that the additive group of $\mathbb{Z}_2[x]/x^2$ is isomorphic to the additive group of $\mathbb{Z}_2 \times \mathbb{Z}_2$, although the rings are not isomorphic.

   (b) Show that the additive group of $\mathbb{Z}_4$ is not isomorphic to the additive group of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2. (a) Show that $\mathbb{Z}_5^\times$ is isomorphic to the additive group of $\mathbb{Z}_4$.

   (b) Show that $\mathbb{Z}_8^\times$ is isomorphic to the additive group of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

   (c) Find a some more small numbers n such that $\mathbb{Z}_n^\times$ is isomorphic to the additive group of $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

3. Show that $\mathbb{R}^\times$ is isomorphic to (the additive group of) $\mathbb{R} \times \mathbb{Z}_2$.

   Optional: Show that $\mathbb{Q}^\times$ is not isomorphic to $\mathbb{Q} \times \mathbb{Z}_2$. Can you give a nice description of $\mathbb{Q}^\times$?

4. Find a subgroup of $\mathbb{C}^\times$ that is isomorphic to $\mathbb{Z}_3$.

5. Recall that $\mathrm{GL}_n(\mathbb{R}) = \mathrm{M}_n(\mathbb{R})^\times$ is the group of invertible $n \times n$ matrices with real entries under matrix multiplication. Let $\mathrm{GL}_n^+(\mathbb{R})$ denote the subgroup of matrices with positive determinant. Show that if n is odd then $\mathrm{GL}_n \cong \mathrm{GL}_n^+ \times \mathbb{Z}_2$.

   Optional: Show that this is not true if n is even. Hint: Consider the *centers* of the two groups:

   $$Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}.$$