Homework 2

Due Wednesday, January 25, 2017

1. Let a group G act on a set S. For a fixed $s \in S$, show that the stabilizer

$$\mathrm{Stab}(s) = \{g \in G : g \cdot s = s\}$$

is a subgroup of G.

2. Let $G = D_4$ be the symmetry group of the square, let $r \in G$ be rotation through $90°$ clockwise, and let s be reflection left-to-right. In lecture we saw that

$$G = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}.$$

Decide which elements on the left are equal to which elements on the right:[1]

rs	sr
r^2s	sr^2
r^3s	sr^3

Do the same with D_5, the symmetry group of the regular pentagon:

rs	sr
r^2s	sr^2
r^3s	sr^3
r^4s	sr^4

[1] In the version of this homework that I handed out, there was a disastrous typo: rs^2 and rs^3 rather than r^2s and r^3s, and similarly for D_5.

3. Label the vertices of the square as shown:

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[d, dash] \\ 4 \arrow[r, dash] & 3 \end{tikzcd}$$

Let D_4 act on the set of 16 *ordered* pairs of vertices:

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & (1,4) \\ \\ (2,1) & (2,2) & (2,3) & (2,4) \\ \\ (3,1) & (3,2) & (3,3) & (3,4) \\ \\ (4,1) & (4,2) & (4,3) & (4,4). \end{array}$$

Indicate the action of r and s by drawing arrows.

There are three orbits. For each one, list the elements; then choose an element and describe its stabilizer.

4. Let G be the additive group of $\mathbb{R}$. Let G act on $S = \mathbb{R}^2$ by

$$\theta \cdot (x, y) = (x \cos\theta - y \sin\theta,\ x \sin\theta + y \cos\theta).$$

Verify that this is a group action, i.e. that

$$0 \cdot (x, y) = (x, y)$$

and that

$$\theta \cdot (\phi \cdot (x, y)) = (\theta + \phi) \cdot (x, y).$$

Describe the orbit and stabilizer of the point $(1, 0)$. Describe all the other orbits.

5. Optional: Let $\mathrm{GL}_n(\mathbb{R})$ act on the set of k-dimensional subspaces of $\mathbb{R}^n$, for a fixed $k \leq n$. Show that the action is transitive, i.e. for any two k-dimensional subspaces $W, W' \subset \mathbb{R}^n$, there is an $A \in \mathrm{GL}_n$ such that $A \cdot W = W'$. Describe the stabilizer of the subspace spanned by the standard basis vectors $e_1, e_2, \ldots, e_k$.