Solutions to Homework 2

1. Let a group G act on a set S. For a fixed $s \in S$, show that the stabilizer

$$\mathrm{Stab}(s) = \{g \in G : g \cdot s = s\}$$

is a subgroup of G.

Solution: Let $g, h \in \mathrm{Stab}(s)$. Then

$$(g \cdot h) \cdot s = g \cdot (h \cdot s) = g \cdot s = s,$$

so $g \cdot h \in \mathrm{Stab}(s)$, and

$$g^{-1} \cdot s = g^{-1} \cdot (g \cdot s) = (g^{-1} \cdot g) \cdot s = 1 \cdot s = s,$$

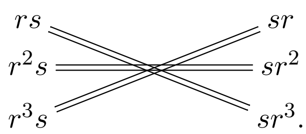
so $g^{-1} \in \mathrm{Stab}(s)$.

2. Let $G = D_4$ be the symmetry group of the square, let $r \in G$ be rotation through $90°$ clockwise, and let s be reflection left-to-right. In lecture we saw that

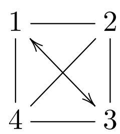
$$G = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}.$$

Decide which elements on the left (below) are equal to which elements on the right. Do the same with D_5, the symmetry group of the regular pentagon.

Solution: For D_4, we find:



Let us justify $rs = sr^3$; the others are similar. We saw in lecture that sr^3 acts by reflecting from northwest to southeast:



For rs, first s sends

$$\begin{tikzcd}
1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[d, dash] \\
4 \arrow[r, dash] & 3
\end{tikzcd}$$

to

$$\begin{tikzcd}
2 \arrow[r, dash] \arrow[d, dash] & 1 \arrow[d, dash] \\
3 \arrow[r, dash] & 4
\end{tikzcd}$$

and then r rotates this to

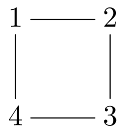
$$\begin{tikzcd}
3 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[d, dash] \\
4 \arrow[r, dash] & 1
\end{tikzcd}$$

which is the same as sr^3.

For D_5, we find:

$$\begin{tikzcd}
rs \arrow[dddrr, equal] & & sr \\
r^2s \arrow[drr, equal] & & sr^2 \\
r^3s \arrow[urr, equal] & & sr^3 \\
r^4s \arrow[uuurr, equal] & & sr^4.
\end{tikzcd}$$

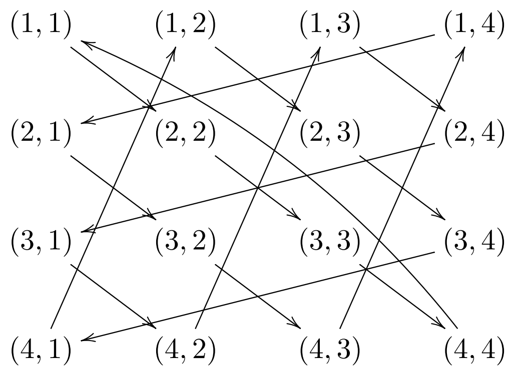
3. Label the vertices of the square as shown:



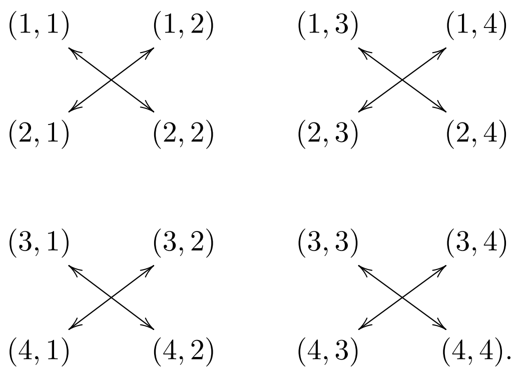
Let D_4 act on the set of 16 *ordered* pairs of vertices. Indicate the action of r and s by drawing arrows.

There are three orbits. For each one, list the elements; then choose an element and describe its stabilizer.

Solution: The rotation r acts as follows:



The reflection s acts as follows:



The three orbits are as follows. First, we have pairs (x, x):

$$\{\ (1,1),\ (2,2),\ (3,3)\ (4,4)\ \}$$

Second, we have pairs (x, y) where x and y are joined by an edge:

$$\{\ (1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,3),\ (4,1)\ \}$$

Third, we have pairs (x, y) where x and y are opposite corners:

$$\{\ (1,3),\ (2,4),\ (3,1),\ (4,2)\ \}$$

The stabilizer of $(1, 1)$ is $\{1, sr\}$. The stabilizer of $(1, 2)$ is just $\{1\}$. The stablizer of $(1, 3)$ is $\{1, sr\}$.

4. Let G be the additive group of $\mathbb{R}$. Let G act on $S = \mathbb{R}^2$ by

$$\theta \cdot (x, y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta).$$

Verify that this is a group action, i.e. that

$$0 \cdot (x, y) = (x, y)$$

and that

$$\theta \cdot (\phi \cdot (x, y)) = (\theta + \phi) \cdot (x, y).$$

Describe the orbit and stabilizer of the point $(1, 0)$. Describe all the other orbits.

Solution: First we have $\cos 0 = 1$ and $\sin 0 = 0$, so

$$0 \cdot (x, y) = (x \cdot 1 - y \cdot 0,\ x \cdot 0 + y \cdot 1) = (x, y).$$

The second claim is an exercise in angle addition formulas:

$$\begin{aligned}
&\theta \cdot (\phi \cdot (x, y)) \\
&= \theta \cdot (x\cos\phi - y\sin\phi,\ x\sin\phi + y\cos\phi) \\
&= (x\cos\phi\cos\theta - y\sin\phi\cos\theta - x\sin\phi\sin\theta - y\cos\phi\sin\theta, \\
&\qquad x\cos\phi\sin\theta - y\sin\phi\sin\theta + x\sin\phi\cos\theta + y\cos\phi\cos\theta) \\
&= (x(\cos\phi\cos\theta - \sin\phi\sin\theta) - y(\sin\phi\cos\theta + \cos\phi\sin\theta), \\
&\qquad x(\cos\phi\sin\theta + \sin\phi\cos\theta) + y(\cos\phi\cos\theta - \sin\phi\sin\theta)) \\
&= (x\cos(\theta + \phi) - y\sin(\theta + \phi),\ x\sin(\theta + \phi) + y\cos(\phi + \theta)) \\
&= (\theta + \phi) \cdot (x, y).
\end{aligned}$$

Thus it is a group action. The orbit of $(1, 0)$ is the unit circle:

$$G \cdot (1, 0) = \{(\cos\theta, \sin\theta) : \theta \in \mathbb{R}\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

The stabilizer of $(1, 0)$ is the subgroup

$$2\pi\mathbb{Z} = \{0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \dots\} \subset \mathbb{R}.$$

The other orbits are cicles of radius r for all $r > 0$, and the origin.

5. Optional: Let $\mathrm{GL}_n(\mathbb{R})$ act on the set of k-dimensional subspaces of $\mathbb{R}^n$, for a fixed $k \leq n$. Show that the action is transitive, i.e. for any two k-dimensional subspaces $W, W' \subset \mathbb{R}^n$, there is an $A \in \mathrm{GL}_n$ such that $A \cdot W = W'$. Describe the stabilizer of the subspace spanned by the standard basis vectors $e_1, e_2, \ldots, e_k$.

   Solution: Part of the problem is to make precise what it means to have $A \cdot W = W'$. What we should mean is this: we take

$$A \cdot W = \{v \in \mathbb{R}^n : v = Aw \text{ for some } w \in W\}.$$

   It is straightforward to check that $A \cdot W$ is a k-dimensional subspace, and that this defines a group action.

   Let U be the subspace spanned by $e_1, e_2, \ldots, e_k$. It is enough to show that for any k-dimensional subspace $W \subset \mathbb{R}^n$ there is a matrix $A \in \mathrm{GL}_n$ such that $A \cdot U = W$. Indeed, if $W, W' \subset \mathbb{R}^n$ are two such subspaces, choose matrices $A, A' \in \mathrm{GL}_n$ such that $A \cdot U = W$ and $A' \cdot U = W'$; then we have $(A'A^{-1}) \cdot W = W'$.

   Now let $W \subset \mathbb{R}^n$ be given. Choose a basis $w_1, \ldots, w_k$ for W, and extend it to a basis $w_1, \ldots, w_k, v_{k+1}, \ldots, v_n$ for $\mathbb{R}^n$. Let A be the matrix whose columns are given by $w_1, \ldots, w_k, v_{k+1}, \ldots, v_n$. Then $A \cdot e_i = w_i$ for $i \leq k$, and v_i for $i > k$. In particular, for any $w \in W$ we can write $w = a_1 w_1 + \cdots + a_k w_k$ for some $a_k \in \mathbb{R}$, and then $A \cdot (a_1 e_1 + \cdots + a_k e_k) = w$, so $W \subset A \cdot U$, and similarly we find that $A \cdot U \subset W$.

   The stabilizer of U consists of matrices of the block form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

   where B is $k \times k$, C is $k \times (n-k)$, and D is $n \times n$. To see this, note again that $A \cdot e_1, \ldots, A \cdot e_k$ are exactly the first k columns of A, so they stay in U if and only if the bottom left block vanishes.