Solutions to Homework 2

1. Let a group \( G \) act on a set \( S \). For a fixed \( s \in S \), show that the stabilizer 
\[
\text{Stab}(s) = \{ g \in G : g \cdot s = s \}
\]
is a subgroup of \( G \).

**Solution:** Let \( g, h \in \text{Stab}(s) \). Then
\[
(g \cdot h) \cdot s = g \cdot (h \cdot s) = g \cdot s = s,
\]
so \( g \cdot h \in \text{Stab}(s) \), and
\[
g^{-1} \cdot s = g^{-1} \cdot (g \cdot s) = (g^{-1} \cdot g) \cdot s = 1 \cdot s = s,
\]
so \( g^{-1} \in \text{Stab}(s) \).

2. Let \( G = D_4 \) be the symmetry group of the square, let \( r \in G \) be rotation through 90° clockwise, and let \( s \) be reflection left-to-right. In lecture we saw that
\[
G = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}.
\]
Decide which elements on the left (below) are equal to which elements on the right. Do the same with \( D_5 \), the symmetry group of the regular pentagon.

**Solution:** For \( D_4 \), we find:

Let us justify \( rs = sr^3 \); the others are similar. We saw in lecture that \( sr^3 \) acts by reflecting from northwest to southeast:
For \( rs \), first \( s \) sends

\[
\begin{array}{c}
1 \\
\downarrow \\
4 \\
\downarrow \\
3
\end{array}
\]

to

\[
\begin{array}{c}
2 \\
\downarrow \\
3 \\
\downarrow \\
1
\end{array}
\]

and then \( r \) rotates this to

\[
\begin{array}{c}
3 \\
\downarrow \\
4 \\
\downarrow \\
1
\end{array}
\]

which is the same as \( sr^3 \).

For \( D_3 \), we find:

\[
\begin{array}{c}
rs \\
\downarrow \\
rs^2 \\
\downarrow \\
rs^3 \\
\downarrow \\
rs^4
\end{array}
\]

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\]

\[
\begin{array}{c}
sr \\
\downarrow \\
sr^2 \\
\downarrow \\
sr^3 \\
\downarrow \\
sr^4
\end{array}
\]
3. Label the vertices of the square as shown:

```
1 ——— 2
|     |
4 ——— 3
```

Let $D_4$ act on the set of 16 ordered pairs of vertices. Indicate the action of $r$ and $s$ by drawing arrows.

There are three orbits. For each one, list the elements; then choose an element and describe its stabilizer.

**Solution:** The rotation $r$ acts as follows:

```
(1,1) ——— (1,2) ——— (1,3) ——— (1,4)
(2,1) ——— (2,2) ——— (2,3) ——— (2,4)
(3,1) ——— (3,2) ——— (3,3) ——— (3,4)
(4,1) ——— (4,2) ——— (4,3) ——— (4,4)
```

The reflection $s$ acts as follows:

```
(1,1) ——— (1,2) ——— (1,3) ——— (1,4)
(2,1) ——— (2,2) ——— (2,3) ——— (2,4)
(3,1) ——— (3,2) ——— (3,3) ——— (3,4)
(4,1) ——— (4,2) ——— (4,3) ——— (4,4)
```

The three orbits are as follows. First, we have pairs $(x, x)$:

\{ (1,1), (2,2), (3,3), (4,4) \}
Second, we have pairs \((x, y)\) where \(x\) and \(y\) are joined by an edge:
\[
\{(1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3), (4, 1)\}
\]

Third, we have pairs \((x, y)\) where \(x\) and \(y\) are opposite corners:
\[
\{(1, 3), (2, 4), (3, 1), (4, 2)\}
\]

The stabilizer of \((1, 1)\) is \(\{1, sr\}\). The stabilizer of \((1, 2)\) is just \(\{1\}\).
The stabilization of \((1, 3)\) is \(\{1, sr\}\).

4. Let \(G\) be the additive group of \(\mathbb{R}\). Let \(G\) act on \(S = \mathbb{R}^2\) by
\[
\theta \cdot (x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta).
\]
Verify that this is a group action, i.e. that
\[
0 \cdot (x, y) = (x, y)
\]
and that
\[
\theta \cdot (\phi \cdot (x, y)) = (\theta + \phi) \cdot (x, y).
\]
Describe the orbit and stabilizer of the point \((1, 0)\). Describe all the other orbits.

**Solution:** First we have \(\cos 0 = 1\) and \(\sin 0 = 0\), so
\[
0 \cdot (x, y) = (x \cdot 1 - y \cdot 0, x \cdot 0 + y \cdot 1) = (x, y).
\]
The second claim is an exercise in angle addition formulas:
\[
\theta \cdot (\phi \cdot (x, y))
\]
\[
= \theta \cdot (x \cos \phi - y \sin \phi, x \sin \phi + y \cos \phi)
\]
\[
= (x \cos \phi \cos \theta - y \sin \phi \cos \theta - x \sin \phi \sin \theta - y \cos \phi \sin \theta,
\]
\[
\quad x \cos \phi \sin \theta - y \sin \phi \sin \theta + x \sin \phi \cos \theta + y \cos \phi \cos \theta)
\]
\[
= (x(\cos \phi \cos \theta - \sin \phi \sin \theta) - y(\sin \phi \cos \theta + \cos \phi \sin \theta),
\]
\[
\quad x(\cos \phi \sin \theta + \sin \phi \cos \theta) + y(\cos \phi \cos \theta - \sin \phi \sin \theta))
\]
\[
= (x \cos(\theta + \phi) - y \sin(\theta + \phi), x \sin(\theta + \phi) + y \cos(\phi + \theta))
\]
\[
= (\theta + \phi) \cdot (x, y).
\]
 Thus it is a group action. The orbit of \((1, 0)\) is the unit circle:
\[
G \cdot (1, 0) = \{(\cos \theta, \sin \theta) : \theta \in \mathbb{R}\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.
\]
The stabilizer of \((1, 0)\) is the subgroup
\[
2\pi \mathbb{Z} = \{0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots\} \subseteq \mathbb{R}.
\]
The other orbits are circles of radius \(r\) for all \(r > 0\), and the origin.
5. Optional: Let \( \text{GL}_n(\mathbb{R}) \) act on the set of \( k \)-dimensional subspaces of \( \mathbb{R}^n \), for a fixed \( k \leq n \). Show that the action is transitive, i.e. for any two \( k \)-dimensional subspaces \( W, W' \subset \mathbb{R}^n \), there is an \( A \in \text{GL}_n \) such that \( A \cdot W = W' \). Describe the stabilizer of the subspace spanned by the standard basis vectors \( e_1, e_2, \ldots, e_k \).

**Solution:** Part of the problem is to make precise what it means to have \( A \cdot W = W' \). What we should mean is this: we take
\[
A \cdot W = \{ v \in \mathbb{R}^n : v = Aw \text{ for some } w \in W \}.
\]
It is straightforward to check that
\( A \cdot W \) is a \( k \)-dimensional subspace, and that this defines a group action.

Let \( U \) be the subspace spanned by \( e_1, e_2, \ldots, e_k \). It is enough to show that for any \( k \)-dimensional subspace \( W \subset \mathbb{R}^n \) there is a matrix \( A \in \text{GL}_n \) such that \( A \cdot U = W \). Indeed, if \( W, W' \subset \mathbb{R}^n \) are two such subspaces, choose matrices \( A, A' \in \text{GL}_n \) such that \( A \cdot U = W \) and \( A' \cdot U = W' \); then we have \( (A' A^{-1}) \cdot W = W' \).

Now let \( W \subset \mathbb{R}^n \) be given. Choose a basis \( w_1, \ldots, w_k \) for \( W \), and extend it to a basis \( w_1, \ldots, w_k, v_{k+1}, \ldots, v_n \) for \( \mathbb{R}^n \). Let \( A \) be the matrix whose columns are given by \( w_1, \ldots, w_k, v_{k+1}, \ldots, v_n \). Then \( A \cdot e_i = w_i \) for \( i \leq k \), and \( v_i \) for \( i > k \). In particular, for any \( w \in W \) we can write \( w = a_1 w_1 + \cdots + a_k w_k \) for some \( a_k \in \mathbb{R} \), and then \( A \cdot (a_1 e_1 + \cdots + a_k e_k) = w \), so \( W \subset A \cdot U \), and similarly we find that \( A \cdot U \subset W \).

The stabilizer of \( U \) consists of matrices of the block form
\[
A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},
\]
where \( B \) is \( k \times k \), \( C \) is \( k \times (n-k) \), and \( D \) is \( n \times n \). To see this, note again that \( A \cdot e_1, \ldots, A \cdot e_k \) are exactly the first \( k \) columns of \( A \), so they stay in \( U \) if and only if the bottom left block vanishes.