Homework 3
Due Wednesday, February 1, 2017

1. Let $G$ be a group. Show that $G$ is Abelian if and only if the map $\varphi : G \to G$ given by $\varphi(g) = g^2$ is a homomorphism.

2. Let $G$ be a group, let $X$ be a set, and let $S_X$ denote the group of bijections $X \to X$ under composition.
   (a) Suppose that $G$ acts on $X$. For $g \in G$, let $l_g : X \to X$ be the map $l_g(x) = g \cdot x$. Show that $l_g \in S_X$, and that the map $G \to S_X$ given by $g \mapsto l_g$ is a homomorphism.
   (b) Conversely, suppose that that $\varphi : G \to S_X$ is a homomorphism. Show that $g \cdot x = \varphi(g)(x)$ defines an action of $G$ on $X$.

3. Let $\sigma$ be the following elements of $S_6$:
   \[\begin{array}{cccccc}
   1 & 2 & 3 & 4 & 5 & 6 \\
   1 & 2 & 3 & 4 & 5 & 6 \\
   \end{array}\]
   Write $\sigma$, $\sigma^2$, and $\sigma^{-1}$ in cycle notation.

4. In $S_6$, let $\sigma = (1 \ 6 \ 2)(3 \ 4)$, $\tau = (3 \ 4 \ 5 \ 6)$.
   Compute $\sigma \cdot \tau$, $\tau \cdot \sigma$, $\sigma^2$, and $\tau^2$.

5. Let $\sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6) \in S_6$. For which integers $n$ is $\sigma^n$ a 6-cycle?

6. Let $G$ be the symmetry group of the regular tetrahedron. In lecture we saw that the action of $G$ on the four vertices induces an isomorphism $G \to S_4$. Draw pictures of the symmetries corresponding to the following elements of $S_4$:
   \( (12) \quad (123) \quad (132) \quad (12)(34) \quad (1234) \)
7. Let the symmetric group $S_3$ act on the vector space $\mathbb{R}^3$ by permuting the three coordinates.

(a) Describe the orbit and stabilizer of the point $(4, 5, 6) \in \mathbb{R}^3$.

(b) Describe the orbit and stabilizer of the point $(4, 5, 5) \in \mathbb{R}^3$.

(c) Optional: Show that $(x, y, z)$ and $(x', y', z')$ lie in the same orbit if and only if

\[
\begin{align*}
    x + y + z &= x' + y' + z', \\
    xy + xz + yz &= x'y' + x'z' + y'z', \\
    xyz &= x'y'z'.
\end{align*}
\]