Homework 4
Due Wednesday, February 8, 2017

1. Label the vertices of a cube as shown:

Let $G$ denote the group of rotations of the cube – no reflections yet. Regard $G$ as a subgroup of $S_{\{a,...,h\}} \cong S_8$ via its action on the eight vertices. For example, the element that rotates the front face by 90 degrees clockwise is $(a\ b\ c\ d)(e\ f\ g\ h)$.

(a) Find the stabilizer of vertex $a$. Conclude that $|G| = 24$.

(b) Let $\varphi: G \rightarrow S_4$ be the homomorphism obtained by letting $G$ act on the long diagonals, labeled as follows:

\[
1 = a - g \quad 2 = b - h \quad 3 = c - e \quad 4 = d - f.
\]

Exhibit elements of $G$ that map to the transpositions $(1\ 2)$, $(1\ 3)$, and $(1\ 4)$. Conclude that $\varphi$ is surjective, and hence also injective, which was not obvious a priori.

(c) Draw a picture of the rotation that maps to $(1\ 2)$ in the previous part. For each of the permutations $(1\ 2\ 3)$, $(1\ 2)(3\ 4)$, and $(1\ 2\ 3\ 4)$ of the long diagonals, write the corresponding permutation of the vertices (in cycle notation), and draw a picture.
2. Optional: We have seen that the symmetry group of the tetrahedron is isomorphic to $S_4$, and that the subgroup of rotations is identified with the subgroup of even permutations $A_4 \subset S_4$ (the alternating group). In the cube above, consider the inscribed tetrahedron $a - c - f - h$, and the subgroup $H \subset G$ that preserves it. Is this the same embedding $A_4 \subset S_4$, or a different one?

3. Optional: Let $\tilde{G}$ be the group of all symmetries of the cube, including reflections etc., and let $G \subset \tilde{G}$ be the subgroup of rotations studied above. Let $\alpha \in \tilde{G}$ denote the antipodal map, which acts on $\mathbb{R}^3$ as $(x, y, z) \rightarrow (-x, -y, -z)$, or on the vertices as

$$(ah)(be)(cf).$$

Show that the map $G \times \mathbb{Z}_2 \rightarrow \tilde{G}$ given by $(g, 0) \mapsto g$ and $(g, 1) \mapsto g \cdot \alpha$ is an isomorphism.

4. Let $F$ be a finite field of order $q$. Show that $x^q = x$ for all $x \in F$. Confirm this explicitly for $F = \mathbb{Z}_5$. Hint: The group of units $F^\times$ has order $q - 1$. The order of an element divides the order of the group.

5. Describe (without proof) all subgroups of $D_4$. Hint: The order of a subgroup divides the order of the group. Further hint: You should find 10 subgroups.

6. Let $G$ be a group with 12 elements

$$\{ 1, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5 \},$$

subject to the relations

$$a^6 = 1 \quad b^2 = a^3 \quad ab = ba^{-1}. \quad \text{(This is very similar to the dihedral group, but we set } b^2 = a^3 \text{ rather than } b^2 = 1.)$$

Let $H$ be the subgroup generated by $b$, which has order 4. Describe its left and right cosets.

7. Let $G$ and $H$ be the following subgroups of $\text{GL}_2(\mathbb{R})$:

$$G = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \quad H = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \quad x > 0.$$ 

Element of $G$ can be represented as a points in the plane. Draw the partition of $G$ into left and right cosets of $H$. 

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