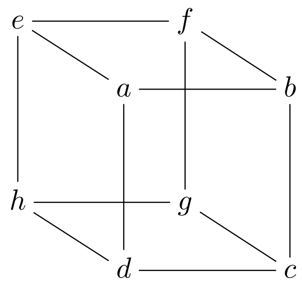
Homework 4

Due Wednesday, February 8, 2017

1. Label the vertices of a cube as shown:



Let G denote the group of *rotations* of the cube – no reflections yet. Regard G as a subgroup of $S_{\{a,\ldots,h\}} \cong S_8$ via its action on the eight vertices. For example, the element that rotates the front face by 90 degrees clockwise is

$$(a\,b\,c\,d)(e\,f\,g\,h).$$

(a) Find the stabilizer of vertex a. Conclude that $|G| = 24$.

(b) Let $\varphi\colon G \to S_4$ be the homomorphism obtained by letting G act on the long diagonals, labeled as follows:

$$1 = a - g \qquad 2 = b - h \qquad 3 = c - e \qquad 4 = d - f.$$

Exhibit elements of G that map to the transpositions $(1\,2)$, $(1\,3)$, and $(1\,4)$. Conclude that φ is surjective, and hence also injective, which was not obvious *a priori*.

(c) Draw a picture of the rotation that maps to $(1\,2)$ in the previous part. For each of the permutations $(1\,2\,3)$, $(1\,2)(3\,4)$, and $(1\,2\,3\,4)$ of the long diagonals, write the corresponding permutation of the vertices (in cycle notation), and draw a picture.

2. Optional: We have seen that the symmetry group of the tetrahedron is isomorphic to S_4, and that the subgroup of rotations is identified with the subgroup of even permutations $A_4 \subset S_4$ (the alternating group). In the cube above, consider the inscribed tetrahedron $a - c - f - h$, and the subgroup $H \subset G$ that preserves it. Is this the same embedding $A_4 \subset S_4$, or a different one?

3. Optional: Let $\tilde{G}$ be the group of all symmetries of the cube, including reflections etc., and let $G \subset \tilde{G}$ be the subgroup of rotations studied above. Let $\alpha \in \tilde{G}$ denote the antipodal map, which acts on $\mathbb{R}^3$ as $(x, y, z) \to (-x, -y, -z)$, or on the vertices as

$$(a\,g)(b\,h)(c\,e)(d\,f).$$

Show that the map $G \times \mathbb{Z}_2 \to \tilde{G}$ given by $(g, 0) \mapsto g$ and $(g, 1) \mapsto g \cdot \alpha$ is an isomorphism.

4. Let F be a finite field of order q. Show that $x^q = x$ for all $x \in F$. Confirm this explicitly for $F = \mathbb{Z}_5$. Hint: The group of units $F^\times$ has order $q - 1$. The order of an element divides the order of the group.

5. Describe (without proof) all subgroups of D_4. Hint: The order of a subgroup divides the order of the group. Further hint: You should find 10 subgroups.

6. Let G be a group with 12 elements

$$\begin{aligned} \{\ &1,\ a,\ \ a^2,\ \ a^3,\ \ a^4,\ \ a^5, \\ &b,\ ba,\ ba^2,\ ba^3,\ ba^4,\ ba^5\ \}, \end{aligned}$$

subject to the relations

$$a^6 = 1 \qquad\qquad b^2 = a^3 \qquad\qquad ab = ba^{-1}.$$

(This is very similar to the dihedral group, but we set $b^2 = a^3$ rather than $b^2 = 1$.) Let H be the subgroup generated by b, which has order 4. Describe its left and right cosets.

7. Let G and H be the following subgroups of $\mathrm{GL}_2(\mathbb{R})$:

$$G = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \qquad\qquad H = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad x > 0.$$

Element of G can be represented as a points in the plane. Draw the partition of G into left and right cosets of H.