1. Label the vertices of a cube as shown:

Let $G$ denote the group of rotations of the cube – no reflections yet. Regard $G$ as a subgroup of $S_{\{a,...,h\}} \cong S_8$ via its action on the eight vertices. For example, the element that rotates the front face by 90 degrees clockwise is $(a b c d)(e f g h)$.

(a) Find the stabilizer of vertex $a$. Conclude that $|G| = 24$.

**Solution:** The stabilizer of vertex $a$ is a copy of $\mathbb{Z}_3$ generated by rotating through 120° about the line joining $a$ and $g$: that is, $(b d e)(c h f)$. The orbit of $a$ is the set of eight vertices. Thus $|G| = 8 \cdot 3 = 24$.

(b) Let $\varphi: G \to S_4$ be the homomorphism obtained by letting $G$ act on the long diagonals, labeled as follows:

\[1 = a - g \quad 2 = b - h \quad 3 = c - e \quad 4 = d - f.\]

Exhibit elements of $G$ that map to the transpositions $(1 2)$, $(1 3)$, and $(1 4)$. Conclude that $\varphi$ is surjective, and hence also injective, which was not obvious *a priori*.

**Solution:** The transposition $(1 2)$ is obtained by rotating through 180° about the line joining the midpoint of edge $a - b$ with the midpoint of edge $g - h$: that is, $(a b)(c e)(d f)(g h)$. 

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The transposition \((1 \, 3)\) is obtained by rotating through 180° about the line joining the midpoint of edge \(a - e\) with the midpoint of edge \(c - g\): that is, \((a \, e)(b \, h)(c \, g)(d \, f)\).

The transposition \((1 \, 4)\) is obtained by rotating through 180° about the line joining the midpoint of edge \(a - d\) with the midpoint of edge \(f - g\): that is, \((a \, d)(b \, h)(c \, e)(f \, g)\).

We have seen in lecture that every element of \(S_4\) is a product of transpositions; and it is also true that every transposition is a product of these three:

\[
\begin{align*}
(2 \, 3) &= (1 \, 2)(1 \, 3)(1 \, 2) \\
(2 \, 4) &= (1 \, 2)(1 \, 4)(1 \, 2) \\
(2 \, 3) &= (1 \, 3)(1 \, 4)(1 \, 3).
\end{align*}
\]

Thus every element of \(S_4\) is a product of elements in the image of \(\varphi\), hence is in the image of \(\varphi\) as we saw in lecture.

(c) Draw a picture of the rotation that maps to \((1 \, 2)\) in the previous part. For each of the permutations \((1 \, 2 \, 3)\), \((1 \, 2)(3 \, 4)\), and \((1 \, 2 \, 3 \, 4)\) of the long diagonals, write the corresponding permutation of the vertices (in cycle notation), and draw a picture.

**Solution:** For \((1 \, 2)\), the picture is:

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\[\text{Picture of rotation (1,2)}\]
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The 3-cycle \((1 \, 2 \, 3)\) is 120° rotation mentioned in part (a), again \((b \, d \, e)(c \, h \, f)\).
The 4-cycle (1 2 3 4) is the 90° rotation of the front face, mentioned before the statement of part (a): (a b c d)(e f g h).

The permutation (1 2)(3 4) is the square of the 4-cycle (1 3 2 4), hence is the 180° rotation about the line joining the left face and the right face: (a h)(b g)(c f)(d e).

2. Optional: We have seen that the symmetry group of the tetrahedron is isomorphic to $S_4$, and that the subgroup of rotations is identified with the subgroup of even permutations $A_4 \subset S_4$ (the alternating group). In the cube above, consider the inscribed tetrahedron a c f h, and the subgroup $H \subset G$ that preserves it. Is this the same embedding $A_4 \subset S_4$, or a different one?

**Solution:** It is the same one. The subgroup $A_4 \subset S_4$ consists of the identity, the eight 3-cycles, and the three products of disjoint 2-cycles; whereas its complement consists of the six transpositions and the six 4-cycles. The 3-cycles correspond to rotating through 120° about a vertex, and the products of disjoint 2-cycles correspond to rotating through 180° about a face, both of which preserve the tetrahedron. (If you want to get into the weeds and say that $(b d e)(c h f)$ and $(a h)(b g)(c f)(d e)$ preserve the set \{a, c, f, h\}, that’s OK too.) Whereas the transpositions correspond to rotating through 180° about the line joining two opposite edges, and the 4-cycles correspond to rotating a face through 90°, both of which take the set \{a, c, f, h\} to \{b, d, e, g\}, which is the other inscribed tetrahedron.
3. Optional: Let $\tilde{G}$ be the group of all symmetries of the cube, including reflections etc., and let $G \subset \tilde{G}$ be the subgroup of rotations studied above. Let $\alpha \in \tilde{G}$ denote the antipodal map, which acts on $\mathbb{R}^3$ as $(x, y, z) \mapsto (-x, -y, -z)$, or on the vertices as $(a, g)(b, h)(c, e)(d, f)$.

Show that the map $G \times \mathbb{Z}_2 \to \tilde{G}$ given by $(g, 0) \mapsto g$ and $(g, 1) \mapsto g \cdot \alpha$ is an isomorphism.

**Solution:** With the orbit-stabilizer theorem we find that $|\tilde{G}| = 48$: now the stabilizer of $a$ also includes reflection across the planes $a - f - g$ and $b - g - h$ and $a - c - e - g$. Thus it is enough to show that the map is an injective (and hence surjective) homomorphism. Call the map $\psi: G \times \mathbb{Z}_2 \to \tilde{G}$.

First we show that $\psi$ is a homomorphism. Observe that $(a, g)(b, h)(c, e)(d, f)$ commutes with the permutations corresponding to $(1, 2)$, $(1, 3)$, and $(1, 4)$ above, hence with any product of them, hence with anything in $G$; or observe that as a map $\mathbb{R}^3 \to \mathbb{R}^3$, the antipodal map $\alpha$ corresponds to

\[
\begin{pmatrix}
-1 & & \\
 & -1 & \\
& & -1
\end{pmatrix},
\]

which commutes with any other matrix. Also observe that $\alpha^2 = 1$. Thus we have

$\psi(g_1, m)\psi(g_2, n) = g_1\alpha^m g_2\alpha^n = g_1g_2\alpha^m\alpha^n = \psi(g_1g_2, m + n)$

For injectivity, one possibility is to observe that the $3 \times 3$ matrix corresponding to any $g \in G$ has determinant 1, while that corresponding to $g\alpha$ has determinant $-1$, so if $g\alpha^n = 1$ then $n = 0$ and $g = 1$.

4. Let $F$ be a finite field of order $q$. Show that $x^q = x$ for all $x \in F$. Confirm this explicitly for $F = \mathbb{Z}_5$. Hint: The group of units $F^\times$ has order $q - 1$. The order of an element divides the order of the group.

**Solution:** Following the hint, we observe that $F^\times = F \setminus \{0\}$ has $q - 1$ elements. Let $x \in F$. If $x = 0$ then clearly $x^q = x$. If $x \neq 0$, let $a$ be the order of $x$ as an element of $F^\times$; in particular $x^a = 1$. But $a | q - 1$, so write $q - 1 = ab$. Then $x^q = x^{ab+1} = (x^a)^b \cdot x = 1^b \cdot x = x$, as desired.
In $\mathbb{Z}_5^\times$, we have

\[
1^5 = 1 \\
2^5 = 32 \equiv 2 \pmod{5} \\
3^5 = 243 \equiv 3 \pmod{5} \\
4^5 = 1024 \equiv 4 \pmod{5}
\]

5. Describe (without proof) all subgroups of $D_4$. Hint: The order of a subgroup divides the order of the group. Further hint: You should find 10 subgroups.

**Solution:** A subgroup may have order 1, 2, 4, or 8.

There is one subgroup of order 1:

\[
\{1\}.
\]

There are five subgroups of order 2, all cyclic:

\[
\langle s \rangle = \{1, s\}, \\
\langle sr \rangle = \{1, sr\}, \\
\langle sr^2 \rangle = \{1, sr^2\}, \\
\langle sr^3 \rangle = \{1, sr^3\}, \\
\langle r^2 \rangle = \{1, r^2\}.
\]

There are three subgroups of order 4: one cyclic

\[
\langle r \rangle = \{1, r, r^2, r^3\},
\]

and two isomorphic to the Klein 4-group:

\[
\langle s, r^2 \rangle = \{1, s, r^2, sr^2\}, \langle sr, r^2 \rangle = \{1, sr, r^2, sr^3\}.
\]

There is one subgroup of order 8, namely the whole group:

\[
\langle s, r \rangle = \{1, s, r, sr, r^2, sr^2, r^3, sr^3\}.
\]
6. Let $G$ be a group with 12 elements

\[
\{ 1, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5 \},
\]

subject to the relations

\[
a^6 = 1 \quad b^2 = a^3 \quad ab = ba^{-1}.
\]

(This is very similar to the dihedral group, but we set $b^2 = a^3$ rather than $b^2 = 1$.) Let $H$ be the subgroup generated by $b$, which has order 4. Describe its left and right cosets.

**Solution:** There are three right cosets:

\[
\begin{align*}
H &= Hb = Ha^3 = Hba^3 = \{1, b, a^3, ba^3\} \\
Ha &= Hba = Ha^4 = Hba^4 = \{a, ba, a^4, ba^4\} \\
Ha^2 &= Hba^2 = Ha^5 = Hba^5 = \{a^2, ba^2, a^5, ba^5\}.
\end{align*}
\]

There are three left cosets:

\[
\begin{align*}
H &= bH = a^3H = ba^3H = \{1, b, a^3, ba^3\} \\
aH &= ba^5H = a^4H = ba^2H = \{a, ba^5, a^4, ba^2\} \\
a^2H &= ba^4H = a^5H = baH = \{a^2, ba^4, a^5, ba\}.
\end{align*}
\]
7. Let $G$ and $H$ be the following subgroups of $\text{GL}_2(\mathbb{R})$:

$$G = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \quad H = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \quad x > 0.$$

Element of $G$ can be represented as a point in the plane. Draw the partition of $G$ into left and right cosets of $H$.

**Solution:** In this case I find it easiest to view the right coset $Hg$ as the orbit of $g \in G$ as $H$ acts by left multiplication. We have

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & ay \\ 0 & 1 \end{pmatrix},$$

so as $(x, y)$ is fixed and $a$ varies we get a ray from the origin through $(x, y)$.

![Diagram of rays](image)

For the left coset $gH$, we let $H$ act by right multiplication. We have

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & y \\ 0 & 1 \end{pmatrix},$$

so as $(x, y)$ is fixed and $a$ varies we get a horizontal ray from the $y$-axis through $(x, y)$.

![Diagram of horizontal rays](image)