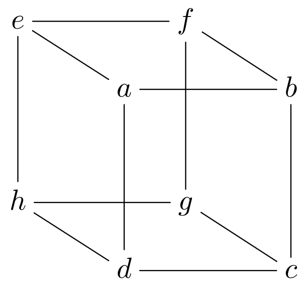
Solutions to Homework 4

1. Label the vertices of a cube as shown:



Let G denote the group of *rotations* of the cube – no reflections yet. Regard G as a subgroup of $S_{\{a,\ldots,h\}} \cong S_8$ via its action on the eight vertices. For example, the element that rotates the front face by 90 degrees clockwise is

$$(a\,b\,c\,d)(e\,f\,g\,h).$$

(a) Find the stabilizer of vertex a. Conclude that $|G| = 24$.

Solution: The stabilizer of vertex a is a copy of $\mathbb{Z}_3$ generated by rotating through 120° about the line joining a and g: that is, $(b\,d\,e)(c\,h\,f)$. The orbit of a is the set of eight vertices. Thus $|G| = 8 \cdot 3 = 24$.

(b) Let $\varphi\colon G \to S_4$ be the homomorphism obtained by letting G act on the long diagonals, labeled as follows:

$$1 = a - g \qquad 2 = b - h \qquad 3 = c - e \qquad 4 = d - f.$$

Exhibit elements of G that map to the transpositions $(1\,2)$, $(1\,3)$, and $(1\,4)$. Conclude that φ is surjective, and hence also injective, which was not obvious *a priori*.

Solution: The transposition $(1\,2)$ is obtained by rotating through 180° about the line joining the midpoint of edge $a - b$ with the midpoint of edge $g - h$: that is, $(a\,b)(c\,e)(d\,f)(g\,h)$.

The transposition $(1\,3)$ is obtained by rotating through $180°$ about the line joining the midpoint of edge $a - e$ with the midpoint of edge $c - g$: that is, $(a\,e)(b\,h)(c\,g)(d\,f)$.

The transposition $(1\,4)$ is obtained by rotating through $180°$ about the line joining the midpoint of edge $a - d$ with the midpoint of edge $f - g$: that is, $(a\,d)(b\,h)(c\,e)(f\,g)$.

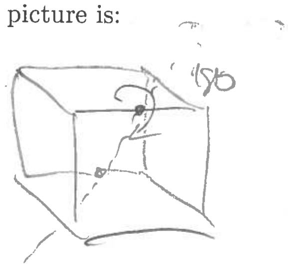
We have seen in lecture that every element of S_4 is a product of transpositions; and it is also true that every transposition is a product of these three:

$$\begin{aligned}(2\,3) &= (1\,2)(1\,3)(1\,2)\\(2\,4) &= (1\,2)(1\,4)(1\,2)\\(2\,3) &= (1\,3)(1\,4)(1\,3).\end{aligned}$$

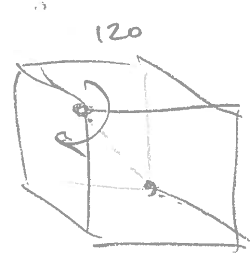
Thus every element of S_4 is a product of elements in the image of φ, hence is in the image of φ as we saw in lecture.

(c) Draw a picture of the rotation that maps to $(1\,2)$ in the previous part. For each of the permutations $(1\,2\,3)$, $(1\,2)(3\,4)$, and $(1\,2\,3\,4)$ of the long diagonals, write the corresponding permutation of the vertices (in cycle notation), and draw a picture.

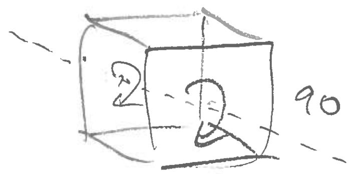
Solution: For $(1\,2)$, the picture is:



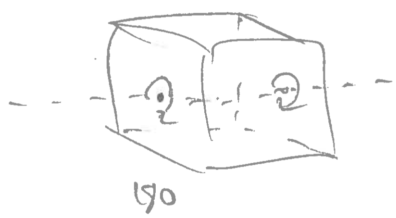
The 3-cycle $(1\,2\,3)$ is $120°$ rotation mentioned in part (a), again $(b\,d\,e)(c\,h\,f)$.



The 4-cycle $(1\,2\,3\,4)$ is the 90° rotation of the front face, mentioned before the statement of part (a): $(a\,b\,c\,d)(e\,f\,g\,h)$.



The permutation $(1\,2)(3\,4)$ is the square of the 4-cycle $(1\,3\,2\,4)$, hence is the 180° rotation about the line joining the left face and the right face: $(a\,h)(b\,g)(c\,f)(d\,e)$.



2. Optional: We have seen that the symmetry group of the tetrahedron is isomorphic to S_4, and that the subgroup of rotations is identified with the subgroup of even permutations $A_4 \subset S_4$ (the alternating group). In the cube above, consider the inscribed tetrahedron $a-c-f-h$, and the subgroup $H \subset G$ that preserves it. Is this the same embedding $A_4 \subset S_4$, or a different one?

   Solution: It is the same one. The subgroup $A_4 \subset S_4$ consists of the identity, the eight 3-cycles, and the three products of disjoint 2-cycles; whereas its complement consists of the six transpositions and the six 4-cycles. The 3-cycles correspond to rotating through 120° about a vertex, and the products of disjoint 2-cycles correspond to rotating through 180° about a face, both of which preserve the tetrahedron. (If you want to get into the weeds and say that $(b\,d\,e)(c\,h\,f)$ and $(a\,h)(b\,g)(c\,f)(d\,e)$ preserve the set $\{a, c, f, h\}$, that's OK too.) Whereas the transpositions correspond to rotating through 180° about the line joining two opposite edges, and the 4-cycles correspond to rotating a face through 90°, both of which take the set $\{a, c, f, h\}$ to $\{b, d, e, g\}$, which is the other inscribed tetrahedron.

3. Optional: Let $\tilde{G}$ be the group of all symmetries of the cube, including reflections etc., and let $G \subset \tilde{G}$ be the subgroup of rotations studied above. Let $\alpha \in \tilde{G}$ denote the antipodal map, which acts on $\mathbb{R}^3$ as $(x, y, z) \to (-x, -y, -z)$, or on the vertices as

$$(a\,g)(b\,h)(c\,e)(d\,f).$$

Show that the map $G \times \mathbb{Z}_2 \to \tilde{G}$ given by $(g, 0) \mapsto g$ and $(g, 1) \mapsto g \cdot \alpha$ is an isomorphism.

Solution: With the orbit-stabilizer theorem we find that $|\tilde{G}| = 48$: now the stabilizer of a also includes reflection across the planes $a-d-f-g$ and $a-b-g-h$ and $a-c-e-g$. Thus it is enough to show that the map is an injective (and hence surjective) homomorphism. Call the map

$$\psi \colon G \times \mathbb{Z}_2 \to \tilde{G}.$$

First we show that ψ is a homomorphism. Observe that $(a\,g)(b\,h)(c\,e)(d\,f)$ commutes with the permutations corresponding to $(1\,2)$, $(1\,3)$, and $(1\,4)$ above, hence with any product of them, hence with anything in G; or observe that as a map $\mathbb{R}^3 \to \mathbb{R}^3$, the antipodal map α corresponds to

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix},$$

which commutes with any other matrix. Also observe that $\alpha^2 = 1$. Thus we have

$$\psi(g_1, m)\psi(g_2, n) = g_1 \alpha^m g_2 \alpha^n = g_1 g_2 \alpha^m \alpha^n = \psi(g_1 g_2, m + n)$$

For injectivity, one possibility is to observe that the 3×3 matrix corresponding to any $g \in G$ has determinant 1, while that corresponding to $g\alpha$ has determinant -1, so if $g\alpha^n = 1$ then $n = 0$ and $g = 1$.

4. Let F be a finite field of order q. Show that $x^q = x$ for all $x \in F$. Confirm this explicitly for $F = \mathbb{Z}_5$. Hint: The group of units $F^\times$ has order $q - 1$. The order of an element divides the order of the group.

Solution: Following the hint, we observe that $F^\times = F \setminus \{0\}$ has $q-1$ elements. Let $x \in F$. If $x = 0$ then clearly $x^q = x$. If $x \neq 0$, let a be the order of x as an element of $F^\times$; in particular $x^a = 1$. But $a \mid q-1$, so write $q - 1 = ab$. Then $x^q = x^{ab+1} = (x^a)^b \cdot x = 1^b \cdot x = x$, as desired.

In $\mathbb{Z}_5^\times$, we have

$$\begin{aligned} 1^5 &= 1 \\ 2^5 &= 32 \equiv 2 \pmod 5 \\ 3^5 &= 243 \equiv 3 \pmod 5 \\ 4^5 &= 1024 \equiv 4 \pmod 5 \end{aligned}$$

5. Describe (without proof) all subgroups of D_4. Hint: The order of a subgroup divides the order of the group. Further hint: You should find 10 subgroups.

   Solution: A subgroup may have order 1, 2, 4, or 8.

   There is one subgroup of order 1:

$$\{1\}.$$

   There are five subgroups of order 2, all cyclic:

$$\begin{aligned} \langle s \rangle &= \{1, s\}, \\ \langle sr \rangle &= \{1, sr\}, \\ \langle sr^2 \rangle &= \{1, sr^2\}, \\ \langle sr^3 \rangle &= \{1, sr^3\}, \\ \langle r^2 \rangle &= \{1, r^2\}. \end{aligned}$$

   There are three subgroups of order 4: one cyclic

$$\langle r \rangle = \{1, r, r^2, r^3\},$$

   and two isomorphic to the Klein 4-group:

$$\langle s, r^2 \rangle = \{1, s, r^2, sr^2\}, \langle sr, r^2 \rangle = \{1, sr, r^2, sr^3\}.$$

   There is one subgroup of order 8, namely the whole group:

$$\langle s, r \rangle = \{1, s, r, sr, r^2, sr^2, r^3, sr^3\}.$$

6. Let G be a group with 12 elements

$$\begin{array}{l} \{ \ 1, \ a, \ \ a^2, \ \ a^3, \ \ a^4, \ \ a^5, \\ \ \ \ b, \ ba, \ ba^2, \ ba^3, \ ba^4, \ ba^5 \ \}, \end{array}$$

subject to the relations

$$a^6 = 1 \qquad\qquad b^2 = a^3 \qquad\qquad ab = ba^{-1}.$$

(This is very similar to the dihedral group, but we set $b^2 = a^3$ rather than $b^2 = 1$.) Let H be the subgroup generated by b, which has order 4. Describe its left and right cosets.

Solution: There are three right cosets:

$$\begin{aligned} H = Hb = Ha^3 = Hba^3 &= \{1, b, a^3, ba^3\} \\ Ha = Hba = Ha^4 = Hba^4 &= \{a, ba, a^4, ba^4\} \\ Ha^2 = Hba^2 = Ha^5 = Hba^5 &= \{a^2, ba^2, a^5, ba^5\}. \end{aligned}$$

There are three left cosets:

$$\begin{aligned} H = bH = a^3H = ba^3H &= \{1, b, a^3, ba^3\} \\ aH = ba^5H = a^4H = ba^2H &= \{a, ba^5, a^4, ba^2\} \\ a^2H = ba^4H = a^5H = baH &= \{a^2, ba^4, a^5, ba\}. \end{aligned}$$

7. Let G and H be the following subgroups of $\mathrm{GL}_2(\mathbb{R})$:

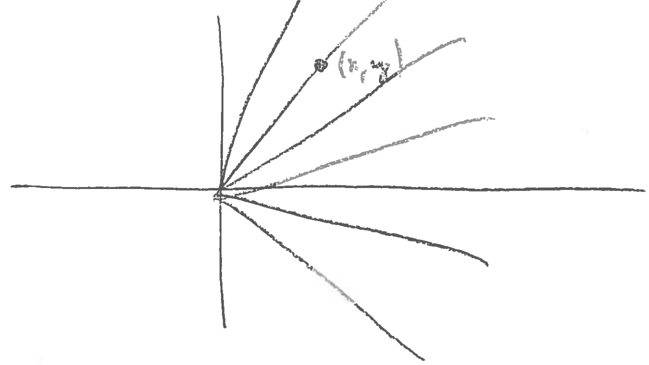
$$G = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \qquad x > 0.$$

Element of G can be represented as a points in the plane. Draw the partition of G into left and right cosets of H.

Solution: In this case I find it easiest to view the right coset Hg as the orbit of $g \in G$ as H acts by left multipliction. We have

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & ay \\ 0 & 1 \end{pmatrix},$$

so as (x, y) is fixed and a varies we get a ray from the origin through (x, y).



For the left coset gH, we let H act by right multiplication. We have

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & y \\ 0 & 1 \end{pmatrix},$$

so as (x, y) is fixed and a varies we get a horizontal ray from the y-axis through (x, y).