Solutions to Homework 5

1. Let G be a group and H a subgroup. For $g, g' \in G$, show that the following are equivalent:

   (a) $gH = g'H$.

   (b) $gH \subset g'H$.

   (c) $gH \cap g'H \neq \emptyset$.

   (d) $g^{-1}g' \in H$.

   (e) $g' \in gH$.

   Warning: Do not assume that G or H is finite.

   Solution: (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are clear.

   For (c) $\Rightarrow$ (d), chose $h, h' \in H$ such that $gh = g'h'$. Multiply on the left by g^{-1} and on the right by h'^{-1} to obtain $g^{-1}g' = hh'^{-1} \in H$.

   For (d) $\Rightarrow$ (e), if $g^{-1}g' = h$ then $g' = gh \in gH$.

   For (e) $\Rightarrow$ (a), write $g' = gh$. For any $g'h' \in g'H$, we have $g'h' = (gh)h' = g(hh') \in gH$; thus $g'H \subset gH$. For any $gh' \in gH$, we have $gh' = (g'h^{-1})h' = g'(h^{-1}h') \in g'H$; thus $gH \subset g'H$.

2. Let G be a group and H a subgroup. Show that the following are equivalent:

   (a) For all $g \in G$ one has $gH = Hg$.

   (b) For all $g \in G$ one has $gH \subset Hg$.

   (c) For all $g \in G$ one has $gHg^{-1} = H$.

   (d) For all $g \in G$ one has $gHg^{-1} \subset H$.

   Notice the difference between the statements

$$\forall g \in G, (gH \subset Hg \Rightarrow gH = Hg)$$

and

$$(\forall g \in G, gH \subset Hg) \Rightarrow (\forall g \in G, gH = Hg).$$

The implication (b) $\Rightarrow$ (a) is the latter. The former holds if H is finite. Optional: Does it hold if H is infinite?

Solution: (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are clear.

For (b) $\Rightarrow$ (a), given $g \in G$ we have both $gH \subset Hg$ and $g^{-1}H \subset Hg^{-1}$; multiplying by g on the left and right we get $Hg \subset gH$, so $gH = Hg$. More carefully, if $g^{-1}H \subset Hg^{-1}$ then for every $h \in H$ there is an $h' \in H$ such that $g^{-1}h = h'g^{-1}$, so $hg = gh'$; thus $Hg \subset gH$, as desired.

The implication (d) $\Rightarrow$ (c) is similar.

For (b) $\Leftrightarrow$ (d), observe that for all $h \in H$ there is an $h' \in H$ with $gh = h'g$, if and only if for all $h \in H$ there is an $h' \in H$ with $ghg^{-1} = h'$.

For the optional part, the answer is no. One options is to let $G \subset \mathrm{GL}_n(\mathbb{R})$ be the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \qquad a \neq 0,$$

and H the subgroup of matrices of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \qquad n \in \mathbb{Z}.$$

Take

$$g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$g \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix},$$

so gHg^{-1} is a proper subgroup of H.

Another option is to let G be the free group on two generates x, y, that is, the group of strings

$$x^{m_1} y^{n_1} x^{m_2} y^{n_2} \cdots x^{m_k} y^{n_k},$$

where k is arbitrary, the exponents $m_i, n_i \in \mathbb{Z}$, and we impose only the most obvious relations: $x^0 = 1$, $x^a x^b = x^{a+b}$, and similarly with

y. Then let H be the subgroup generated by $x^m y^n x^{-m}$, where $m \geq 0$ and n is arbitrary: that is, elements where, as we read from left to right and keep a running sum of the exponents of x, the sum is always non-negative, and returns to zero by the end. Then $xHx^{-1} \subset H$, but $y \notin xHx^{-1}$, because then we would have $x^{-1}yx \in H$.

3. (a) Let a group G act on a set X (on the left), and let $H \subset G$ be the stabilizer of a point $x \in X$. Show that for $g \in G$, the stabilizer of gx is gHg^{-1}.

   Solution: First, for any $ghg^{-1} \in gHg^{-1}$, we have $ghg^{-1} \cdot gx = ghx = gx$, so ghg^{-1} is in the stabilizer of gx.

   For the reverse inclusion, suppose that k is in the stabilizer of gx. Then $kgx = gx$, so $g^{-1}kgx = x$, so $g^{-1}kg \in H$. Writing $g^{-1}kg = h$, we then have $k = ghg^{-1} \in gHg^{-1}$.

   Conversely, for $ghg^{-1} \in gHg^{-1}$, we have $ghg^{-1} \cdot gx = ghx = gx$, so ghg^{-1} is in the stabilizer.

   (b) Give an explicit example where $G = D_4$.

   Solution: Let $G = D_4$ act on the four vertices of the square, numbered as shown,

$$\begin{tikzcd} 1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[d, dash] \\ 4 \arrow[r, dash] & 3 \end{tikzcd}$$

   and let r act by rotating clockwise through $90°$, and s act by reflecting left-to-right. We find that

$$\begin{aligned} \mathrm{Stab}(1) &= \{1, sr\} \\ \mathrm{Stab}(2) &= \{1, sr^3\} \end{aligned}$$

   We have $r \cdot 1 = 2$, and $r \cdot sr \cdot r^{-1} = rs = sr^3$, so indeed $r\,\mathrm{Stab}(1)r^{-1} = \mathrm{Stab}(2)$.

4. Let G be a group of order 8, and suppose that $g^2 = 1$ for all $g \in G$. Show that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

   Solution: From the midterm we know that G is Abelian. Since G has only one element of order 1, it has seven elements of order 2. Let $x, y \in G$ be two distinct elements. Then $xy = yx$ is distinct from 1, x, y: indeed, if $xy = 1$ then $x = xy^2 = y$; if $xy = x$ then $y = 1$, and if $xy = y$ then $x = 1$.

Let $z \in G$ be distinct from $\{1, x, y, xy\}$. Then xz, yz, and xyz are distinct from those four, from z, and from one another by a similar argument.

Thus the map $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to G$ given by $(a, b, c) \mapsto x^a y^b z^c$ is an isomorphism.

5. (a) Let $\varphi\colon G \to H$ be a surjective group homomorphism, and let $K = \ker\varphi$. Show that for all $h \in H$, the fiber

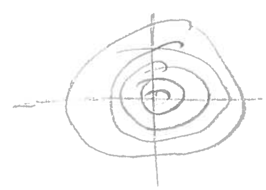
$$\varphi^{-1}(h) = \{g \in G : \varphi(g) = h\}$$

is a coset of K. (Left or right doesn't matter, because K is normal. As always, beware that φ^{-1} is not a map $H \to G$.)

Solution: Since φ is surjective, we can choose a $g \in \varphi^{-1}(h)$. I claim that $gK = \varphi^{-1}(h)$. First, if $gk \in gK$ then $\varphi(gk) = \varphi(g)\varphi(k) = h \cdot 1 = h$, so $gk \in \varphi^{-1}(h)$. Second, if $g' \in \varphi^{-1}(h)$ then $\varphi(g^{-1}g') = \varphi(g)^{-1}\varphi(g') = h^{-1}h = 1$, so $g^{-1}g' \in K$, so $g' \in gK$ by problem 1.

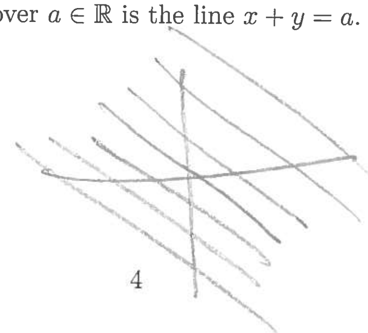
(b) For the homomorphism $\varphi\colon \mathbb{C}^\times \to \mathbb{R}$ given by $\varphi(z) = \log|z|$, describe the kernel and its cosets geometrically.

Solution: They are circles centered at the (missing) origin. The coset of $z \in \mathbb{C}^\times$ is the circle of radius $|z|$. Alternatively, the fiber of φ over $x \in \mathbb{R}$ is the circle of radius e^x.



(c) For the homomorphism $\varphi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $\varphi(x, y) = x + y$, describe the kernel and its cosets geometrically.

Solution: They are lines of slope -1. The coset of a point (x, y) is the line of slope -1 passing through the point. Alternatively, the fiber of φ over $a \in \mathbb{R}$ is the line $x + y = a$.



6. Optional: Show that

(a) $D_6 \cong D_3 \times \mathbb{Z}_2$.

Solution: Define a map $\varphi\colon D_3 \times \mathbb{Z}_2 \to D_6$ by

$$\varphi(s^a r^b, c) = s^a r^{2b+3c}.$$

This is well-defined because $r^6 = 1$ in D_6. To see that φ is a homomorphism, we consider two cases:

$$\begin{aligned}
\varphi(s^a r^{b_1}, c_1) \cdot \varphi(r^{b_2}, c_2) &= s^a r^{2b_1+3c_1} r^{2b_2+3c_2} \\
&= s^a r^{2(b_1+b_2)+3(c_1+c_2)} \\
&= \varphi(s^a r^{b_1+b_2}, c_1 + c_2)
\end{aligned}$$

$$\begin{aligned}
\varphi(s^a r^{b_1}, c_1) \cdot \varphi(s r^{b_2}, c_2) &= s^a r^{2b_1+3c_1} s r^{2b_2+3c_2} \\
&= s^a s r^{-(2b_1+3c_1)} r^{2b_2+3c_2} \\
&= s^{a+1} r^{2(-b_1+b_2)+3(-c_1+c_2)} \\
&= \varphi(s^{a+1} r^{-b_1+b_2}, -c_1 + c_2) \\
&= \varphi(s^a s r^{-b_1} r^{b_2}, c_1 + c_2) \\
&= \varphi(s^a r^{b_1} s r^{b_2}, c_1 + c_2).
\end{aligned}$$

To see that φ is surjective, we observe that any integer can be written in the form $2b + 3c$ for some $b, c \in \mathbb{Z}$ because 3 is odd.

(b) $D_8 \ncong D_4 \times \mathbb{Z}_2$.

Solution: First we establish that for $n \geq 2$, the center

$$Z(D_{2n}) = \{1, r^n\}.$$

For a rotation $r^m \in D_{2n}$, we have $r^m \in Z(D_n)$ if and only if $sr^m = r^m s$, which is true if and only if $sr^m = sr^{-m}$, which is true if and only if $m \equiv 0$ or $n \pmod{2n}$. For a reflection $sr^m \in D_{2n}$, we have $sr^m \cdot r = r \cdot sr^m = sr^{m-1}$, so if $sr^m \in Z(D_{2n})$ then $m + 1 \equiv m - 1 \pmod{2n}$, which is impossible because $2n \geq 4$. Thus the center of D_8 has order 2. But

$$Z(D_4 \times \mathbb{Z}_2) \cong Z(D_4) \times Z(\mathbb{Z}_2),$$

as we saw on the optional part of homework 1, problem 5, and this has order 4.

(c) Generalize to D_{2n}.

Solution: If n is odd then the map $\varphi\colon D_n \times \mathbb{Z}_2 \to D_{2n}$ given by

$$\varphi(s^a r^b, c) = s^a r^{2b+nc}$$

is an isomorphism. If n is even then $Z(D_{2n})$ has order 2 while $Z(D_n \times \mathbb{Z}_2)$ has order 4.

Geometrically, we can consider an n-gon inscribed in a $2n$-gon; then the antipodal map (= rotation through 180°) preserves the n-gon if n is even, and does not preserve it if n is odd.