Solutions to Homework 7

1. In lecture we showed that every group of order 9 is Abelian. Write the proof in your own words, according to your own taste.

   Optional: Do p^2 rather than 9.

   Solution: Here's how I'd write it formally.

   First we argue that if G is a group of order p^m, where p is prime and $m \geq 2$, then the center $Z = Z(G)$ has order p^k for some k between 1 and m. Since $|Z|$ divides $|G|$, we know that $|Z| = p^k$ for some k between 0 and m; so it is enough to show that p divides $|Z|$.

   The order of a conjugacy class divides the order of the group, and is strictly less than the order of the group (since the identity is in a conjugacy class by itself), so a conjugacy class may have order $1, p, p^2, \ldots, p^{m-1}$. Let a_l be the number of conjugacy classes of order p^l for $l = 0, 1, \ldots, m-1$. As we observed in lecture, an element is in $Z(G)$ if and only if its conjugacy class has order 1, so $|Z| = a_0$. Now we have

   $$|G| = p^m = a_0 + a_1 p + a_2 p^2 + \cdots + a_m p^{m-1},$$

   so a_0 is a multiple of p, as desired.

   Next we argue that if G is a group of order p^2 then G is Abelian. We have seen that $|Z| = p$ or p^2. If $|Z| = p^2$ then $Z = G$, so we are done. If $|Z| = p$ then we will derive a contradiction. Indeed, G/Z would be a group of order p, hence would be isomorphic to $\mathbb{Z}_p$. Choose a $g \in G$ such that gZ generates G/Z: that is, for all $x \in G$ we have $xZ = (gZ)^i = g^i Z$ for some integer i. Thus $x = g^i z$ for some $z \in Z$. For any other $y \in G$ we can write $y = g^j w$ for some integer j and $w \in Z$. Thus,

   $$xy = g^i z g^j w = g^j w g^i z = yx,$$

   because z and w commute with g and g^i commutes with g^j. Thus $x, y \in Z$, so $G = Z$, contradicting our assumption that $|Z| = p$.

2. Let G be the group of order 12 from Homework 4, problem 6:

$$\begin{array}{l}\{\ 1,\ a,\ \ a^2,\ \ a^3,\ \ a^4,\ \ a^5, \\ \quad b,\ ba,\ ba^2,\ ba^3,\ ba^4,\ ba^5\ \},\end{array}$$

subject to the relations

$$a^6 = 1 \qquad\qquad b^2 = a^3 \qquad\qquad ab = ba^{-1}.$$

Describe the conjugacy classes and write the class equation for G.

Solution: Multiplying the third relation by a^{-1} on the right we find that

$$aba^{-1} = ba^{-2} = ba^4.$$

Since conjugating by a is a group homomorphism, we find:

g	1	a	a^2	a^3	a^4	a^5	b	ba	ba^2	ba^3	ba^4	ba^5
aga^{-1}	1	a	a^2	a^3	a^4	a^5	ba^4	ba^5	b	ba	ba^2	ba^3

Multiplying the third relation by b^{-1} on the right we find that

$$ba^{-1}b^{-1} = a,$$

so, because conjugating by b is a group homomorphism,

$$bab^{-1} = a^{-1} = a^5.$$

Thus:

g	1	a	a^2	a^3	a^4	a^5	b	ba	ba^2	ba^3	ba^4	ba^5
bgb^{-1}	1	a^5	a^4	a^3	a^2	a^1	b	ba^5	ba^4	ba^3	ba^2	ba

Thus the conjugacy classes are as follows:

$$\{1\},\ \{a, a^5\},\ \{a^2, a^4\},\ \{a^3\},\ \{b, ba^2, ba^4\}, \{ba, ba^3, ba^5\}.$$

The class equation is

$$12 = 2 + 2 + 2 + 3 + 3.$$

3. Let $\sigma = (1\,2\,3\,4\,5) \in S_5$.

   (a) Find a $\tau \in S_5$ such that $\tau\sigma\tau^{-1} = \sigma^2$.
   Solution: We compute $\sigma^2 = (1\,3\,5\,2\,4)$. There are five possibilities for τ, all of them four-cycles:

$$(2\,3\,5\,4) \qquad (1\,3\,2\,5) \qquad (1\,5\,3\,4) \qquad (1\,2\,4\,3) \qquad (1\,4\,5\,2)$$

(b) Find a $\tau \in S_5$ such that $\tau\sigma\tau^{-1} = \sigma^3$.

Solution: We compute $\sigma^3 = (1\,4\,2\,5\,3)$. There are five possibilities for τ, all of them four-cycles:

$$(2\,4\,5\,3) \qquad (1\,4\,3\,5) \qquad (1\,2\,5\,4) \qquad (1\,5\,2\,3) \qquad (1\,3\,4\,2)$$

Indeed, these are inverses of the answers to part (a), because if $\tau\sigma\tau^{-1} = \sigma^2$ then $\sigma = \tau^{-1}\sigma^2\tau$, so $\sigma^3 = \tau^{-1}\sigma^6\tau = \tau^{-1}\sigma\tau$.

(c) Find a $\tau \in S_5$ such that $\tau\sigma\tau^{-1} = \sigma^4$.

Solution: We compute $\sigma^4 = (1\,5\,4\,3\,2)$. There are five possibilities for τ, all of them products of disjoint transpositions:

$$(2\,5)(3\,4) \qquad (1\,5)(2\,4) \qquad (1\,4)(2\,3) \qquad (1\,3)(4\,5) \qquad (1\,2)(3\,5)$$

Indeed, these are the squares of the answers to part (a), because if $\tau\sigma\tau^{-1} = \sigma^2$ then $\tau^2\sigma\tau^{-2} = \tau\sigma^2\tau^{-1} = \sigma^4$.

4. Very optional: Let $G = \mathrm{GL}_2(\mathbb{C})$. Show that every $A \in G$ is conjugate to exactly one of the following:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},\ \lambda \neq \mu \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Solution: First we argue that these three are not conjugate to one another: In the first case, the characteristic polynomial has distinct roots, whereas in the second and third cases, it has a double root. In the second case, A is in the center of G, hence is not conjugate to anything but itself.

Now we argue that any $A \in \mathrm{GL}_2(\mathbb{C})$ is conjgate to one of these three. Write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and consider the characteristic polynomial

$$\chi(t) = \det(A - tI) = t^2 - (a+d)t + (ad - bc).$$

Because $\mathbb{C}$ is algebraically closed, this has either two distinct roots in $\mathbb{C}$, or a double root. If λ is a root of χ then there is a non-zero $v \in \mathbb{C}^2$ such that $Av = \lambda v$. (In this 2×2 case we can be very explicit if we want to: if $a \neq \lambda$ or $b \neq 0$ then we can take

$$v = \begin{pmatrix} -b \\ a - \lambda \end{pmatrix},$$

if $d \neq \lambda$ or $c \neq 0$ then we can take

$$v = \begin{pmatrix} d - \lambda \\ -c \end{pmatrix},$$

and if $a = d = \lambda$ and $b = c = 0$ then we can take any non-zero v.)

If χ has two distinct roots $\lambda \neq \mu$, let $v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ be a λ-eigenvector, and let $w = \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right)$ be a μ-eigenvector. Then v and w are linearly independent, so the matrix

$$B := \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

is invertible. Then we have

$$AB = \begin{pmatrix} \lambda v_1 & \mu w_1 \\ \lambda v_2 & \mu w_2 \end{pmatrix} = B \cdot \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

so

$$B^{-1}AB = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

If χ has a double root λ, let v be a λ-eigenvector, and let w be any vector linearly independent from v. Then we have

$$AB = \begin{pmatrix} \lambda v_1 & x \\ \lambda v_2 & y \end{pmatrix} = B \cdot \begin{pmatrix} \lambda & z \\ 0 & w \end{pmatrix}$$

for some $x, y, z, w \in \mathbb{C}$, so

$$B^{-1}AB = \begin{pmatrix} \lambda & z \\ 0 & w \end{pmatrix}.$$

Because the characteristic polynomial is invariant under conjugation, we find that $w = \lambda$. If $z = 0$ then $A = \lambda I$. If $z \neq 0$ then observe that

$$\begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & z \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$