

Solutions Homework 8

1. Let D_6 be the dihedral group of the hexagon, which has $12 = 2^2 \cdot 3$ elements.

- (a) List all Sylow 2-subgroups of D_6 , i.e. subgroups of order $2^2 = 4$. Confirm that they are all conjugate to one another, and that the number n_2 of such subgroups satisfies $n_2 \equiv 1 \pmod{2}$ and $n_2 \mid 3$.

Solution: Since G has no element of order 4, every subgroup of order 4 must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by two commuting elements of order 2. The elements of order 2 are r^3 , which commutes with everything, and the reflections sr^i for $i = 0, 1, \dots, 5$, which do not commute with one another. Thus we find three Sylow 2-subgroups:

$$\begin{aligned}\langle r^3, s \rangle &= \langle r^3, sr^3 \rangle = \{1, r^3, s, sr^3\} \\ \langle r^3, sr \rangle &= \langle r^3, sr^4 \rangle = \{1, r^3, sr, sr^4\} \\ \langle r^3, sr^2 \rangle &= \langle r^3, sr^5 \rangle = \{1, r^3, sr^2, sr^5\}.\end{aligned}$$

The first is conjugate to the second because $rsr^{-1} = sr^{-2} = sr^4$, and to the third because $r^2sr^{-2} = sr^{-4} = sr^2$. Finally we have $n_2 = 3$, which is odd and divides 3.

- (b) List all Sylow 3-subgroups of D_6 . Confirm that they are all conjugate to one another, and that $n_3 \equiv 1 \pmod{3}$ and $n_3 \mid 4$.

Solution: A Sylow 3-subgroup of D_6 has order 3, hence is a cyclic subgroup generated by an element of order 3. There are two of these, namely r^2 and r^4 , and they generate the same subgroup:

$$\langle r^2 \rangle = \langle r^4 \rangle = \{1, r^2, r^4\}.$$

Of course it is conjugate to itself. We have $n_3 = 1$, which is congruent to 1 mod 3 and divides 4.

2. Let G be the group of order 12 from Homework 4, problem 6 and Homework 7, problem 2:

$$G = \{ 1, a, a^2, a^3, a^4, a^5, b, ba, ba^2, ba^3, ba^4, ba^5 \},$$

subject to the relations

$$a^6 = 1 \qquad b^2 = a^3 \qquad ab = ba^{-1}.$$

- (a) List all Sylow 2-subgroups of G . Confirm that they are all conjugate to one another, and that $n_2 \equiv 1 \pmod{2}$ and $n_2 \mid 3$.

Solution: A Sylow 2-subgroup of G has order 4, hence is isomorphic to \mathbb{Z}_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since G has only one element of order 2, namely a^3 , it cannot contain a copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence every Sylow 2-subgroup is cyclic, generated by an element of order 4. There are six of these, namely ba^i for $i = 0, 1, \dots, 5$. For each one we find that

$$(ba^i)^2 = ba^i \cdot ba^i = b \cdot ba^{-i} \cdot a^i = b^2 = a^3,$$

so

$$(ba^i)^3 = ba^i \cdot (ba^i)^2 = ba^i \cdot a^3 = ba^{i+3}$$

and

$$(ba^i)^4 = ((ba^i)^2)^2 = (a^3)^2 = a^6 = 1.$$

They generate three distinct subgroups:

$$\begin{aligned} \langle b \rangle &= \langle ba^3 \rangle = \{1, b, a^3, ba^3\} \\ \langle ba \rangle &= \langle ba^4 \rangle = \{1, ba, a^3, ba^4\} \\ \langle ba^2 \rangle &= \langle ba^5 \rangle = \{1, ba^2, a^3, ba^5\}. \end{aligned}$$

The first is conjugate to the second, because $aba^{-1} = ba^{-2} = ba^4$, and to the third because $a^2ba^{-2} = ba^{-4} = ba^2$. Finally we have $n_2 = 3$, which is odd and divides 3.

- (b) List all Sylow 3-subgroups of G . Confirm that they are all conjugate to one another, and that $n_3 \equiv 1 \pmod{3}$ and $n_3 \mid 4$.

Solution: This is similar to 1(b): there is one Sylow 3-subgroup, namely

$$\{1, a^2, a^4\}.$$

3. Optional: List all Sylow 2- and 3-subgroups of S_4 . Exhibit two subgroups of order 4 (not 8!) that are not isomorphic, hence not conjugate.

Solution: We have $|S_4| = 4! = 24 = 2^3 \cdot 3$.

Let's do Sylow 3-subgroups first, since those are easier: they are generated by elements of order 3, i.e. 3-cycles; a 3-cycle and its inverse generate the same subgroup; thus we get $\binom{4}{3} = 4$ subgroups:

$$\begin{aligned} &\{1, (123), (132)\}, \\ &\{1, (124), (142)\}, \\ &\{1, (134), (143)\}, \\ &\{1, (234), (243)\}. \end{aligned}$$

We have $n_3 = 4$, which satisfies $4 \equiv 1 \pmod{3}$ and $4 \mid 8$.

Next let's do Sylow 2-subgroups, which have order 8. I can think of three of them off the top of my head: for each of the following ways of labelling the vertices of the square, we get an embedding $D_4 \hookrightarrow S_4$:

$$\begin{array}{ccc} \begin{array}{cc} 1 & \text{---} & 2 \\ | & & | \\ 4 & \text{---} & 3 \end{array} & \begin{array}{cc} 1 & \text{---} & 2 \\ | & & | \\ 3 & \text{---} & 4 \end{array} & \begin{array}{cc} 1 & \text{---} & 3 \\ | & & | \\ 4 & \text{---} & 2 \end{array} \end{array}$$

Any other labelling differs from one of these by a symmetry of the square. As we discussed in lecture (but you might want to elaborate on in your solutions), the embedding from the first labelling is conjugate to the embedding from the second labelling via (34) , and to the embedding from the third by (23) .

How should we argue that these are all the subgroups of order 8? If we believe the third Sylow theorem then we know that $n_2 \mid 3$, so we're done. Another possibility is to argue that every subgroup of order 8 contains the normal subgroup

$$N = \{1, (12)(34), (13)(24), (14)(23)\},$$

then argue that $S_4/N \cong S_3$, then say that subgroups $H \subset S_4$ of order 8 correspond to subgroups $H/N \subset S_4/N \cong S_3$ of order 2, and you know what those are; but this is pretty fancy. You could always write down the orders of all the elements in S_4 and try to analyze it that way, but the argument becomes very fiddly.

For the last part, about subgroups of order 4, let me just list them all. We have the normal subgroup N given above, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are three other subgroups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by pairs of disjoint transpositions:

$$\begin{aligned}\langle(12), (34)\rangle &= \{1, (12), (34), (12)(34)\}, \\ \langle(13), (24)\rangle &= \{1, (13), (24), (13)(24)\}, \\ \langle(14), (23)\rangle &= \{1, (14), (23), (14)(23)\}.\end{aligned}$$

These are not normal: the first is conjugate to the second via (23) , and to the third via (24) . Finally there are the subgroups isomorphic to \mathbb{Z}_4 , generated by 4-cycles:

$$\begin{aligned}\langle(1234)\rangle &= \{1, (1234), (13)(24), (1432)\}, \\ \langle(1243)\rangle &= \{1, (1243), (14)(23), (1342)\}, \\ \langle(1324)\rangle &= \{1, (1324), (12)(34), (1423)\}.\end{aligned}$$

These are conjugate to one another because any two 4-cycles are conjugate. Thus the subgroups of order 4 are much more diverse than the subgroups of order 8.

4. (a) Let G be a group and let $K \subset H \subset G$ be subgroups. Show that if K is normal in G then K is normal in H .

Solution: Let $h \in H$ be given. Because K is normal in G and $h \in G$, we have $hKh^{-1} = K$.

- (b) Let $G = D_4$, let $H = \{1, s, r^2, sr^2\}$, and let $K = \{1, s\}$. Show that H is normal in G , and K is normal in H , but K is not normal in G .

Solution: The index of H in G is $\frac{|G|}{|H|} = \frac{8}{4} = 2$, so H is normal in G . The index of K in H is $\frac{|H|}{|K|} = \frac{4}{2} = 2$, so K is normal in H . But $rKr^{-1} = \{1, sr^2\} \neq K$, so K is not normal in G .