Homework 9

Due Wednesday, March 15, 2017

1. In lecture we showed that every group of order $15 = 3 \cdot 5$ is cyclic: we must have $n_3 = 1$ and $n_5 = 1$, so there aren't enough elements of orders 1, 3, and 5 to fill out the group, and there must be an element of order 15.

   (a) Write out the details.

   (b) For which other pairs of primes $p < q$ does the same argument show that every group of order pq is cyclic?

2. Show that a group of order $14 = 2 \cdot 7$ is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_7 \cong \mathbb{Z}_{14}$, or to D_7.

   Hint: Emulate what we did in lecture with groups of order 10.

3. Let G be a group of order $20 = 2^2 \cdot 5$. Show that G has exactly four elements of order 5. Hint: Think about n_5.

   Optional: Show if a Sylow 2-subgroup is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ then G is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$, or to $\mathbb{Z}_2 \times D_5$.

   Very optional: See what happens if a Sylow 2-subgroup is isomorphic to $\mathbb{Z}_4$.

(Continued on back.)

4. Let $k = \mathbb{Z}_3$, thought of as a field with elements $\{-1, 0, 1\}$. Let T be the group of 2×2 upper triangular matrices with entries in k:

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \; a, b, c \in k, a \neq 0, c \neq 0 \right\}.$$

(a) Show that T is not Abelian, and $|T| = 12$.

We know three non-Abelian groups of order 12: D_6, A_4, and

$$G = \langle a, b \mid a^6 = 1, b^2 = a^3, ab = ba^{-1} \rangle.$$

(b) Show that T has $n_3 = 1$, as follows. Consider the map $\varphi \colon T \to k^\times \times k^\times$ given by

$$\varphi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (a, c).$$

Show that φ is a homomorphism with $|\ker \varphi| = 3$. Thus $\ker \varphi$ is a Sylow 3-subgroup and is normal.

Conclude that $T \ncong A_4$. (What was n_3 for A_4?)

(c) Show that T has a Sylow 2-subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (Hint: Take $b = 0$.)

Conclude that $T \ncong G$. (What were the Sylow 2-subgroups of G?)

(d) So we must have $T \cong D_6$. Find an element $r \in T$ with $r^6 = 1$, and an element $s \in T$ with $s^2 = 1$ and $rs = sr^{-1}$.

(e) Let T act on the set of column vectors $V = k^2$. Show that there are three orbits:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} * \\ \pm 1 \end{pmatrix}$$

Write the six elements of the last orbit in a hexagon, in such a way that your matrix r from part (d) acts as a rotation, and s acts a reflection.

Hint: Write $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ at the top, then multiply by r over and over.