Solutions to Homework 9

1. In lecture we showed that every group of order $15 = 3 \cdot 5$ is cyclic: we must have $n_3 = 1$ and $n_5 = 1$, so there aren't enough elements of orders 1, 3, and 5 to fill out the group, and there must be an element of order 15.

   (a) Write out the details.

   Solution: By Sylow's third theorem we have $n_3 \mid 5$, so $n_3 = 1$ or 5, and $n_3 \equiv 1 \pmod 3$, so $n = 1$. Similarly, $n_5 \mid 3$, so $n_5 = 1$ or 3, and $n_5 \equiv 1 \pmod 5$, so $n_5 = 1$.

   The order of any element divides 15, hence equals 1, 3, 5, or 15. There is only one element of order 1, the identity. Every element of order 3 generates a subgroup of order 3, which is a Sylow 3-subgroup; since there is only one such subgroup, there are only two elements of order 3. Similarly, there are only four elements of order 5. Since $1 + 2 + 4 < 15$, there must be an element of order 15. Thus the group is cyclic.

   (b) For which other pairs of primes $p < q$ does the same argument show that every group of order pq is cyclic?

   Solution: The same argument works unless $q \equiv 1 \pmod p$, as follows.

   We have $n_p \mid q$, so $n_p = 1$ or q, and $n_p \equiv 1 \pmod p$. If $q \not\equiv 1 \pmod p$ then we can conclude that $n_p = 1$. Next, we have $n_q \mid p$, so $n_q = 1$ or p, and $n_q \equiv 1 \pmod q$; since $p < q$ we can say conclude that $n_q = 1$.

   As before we find that there is one element of order 1, $p - 1$ elements of order p, and $q - 1$ elements of order q. We want to say that

   $$1 + (p - 1) + (q - 1) < pq,$$

   or equivalently,

   $$(p - 1)(q - 1) > 0.$$

Since $p \geq 2$ and $q \geq 2$, this is true. Thus there must be an element of order pq, so the group is cyclic.

Among groups of order less than 100, this argument applies to groups of order 15, 33, 35, 51, 65, 69, 77, 85, 87, 91, and 95.

2. Show that a group of order $14 = 2 \cdot 7$ is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_7 \cong \mathbb{Z}_{14}$, or to D_7.

   Solution: Let H be a Sylow 7-subgroup, which must have order 7, and let x be a generator of H. We have $n_7 \mid 2$, so $n_7 = 1$ or 2, and $n_7 \equiv 1 \pmod 7$, so $n_7 = 1$, so H is normal.

   Let K be a Sylow 2-subgroup, which must have order 2, and let y be a generator of K.

   Then x and y generate the group G, as follows. We have $H \cap K = \{1\}$, because $|H \cap K|$ must divide both $|H| = 7$ and $|K| = 2$. Thus $y \notin H$, so $Hy \neq H$, so by a counting argument we have $G = H \cup Hy$, so every element is of the form $x^i y^j$ for some $i \in \{0, 1, \dots, 6\}$ and $j \in \{0, 1\}$.

   Because H is normal, we have $yxy^{-1} = x^n$ for some $n \in \mathbb{Z}$. Because $y^2 = 1$, we have $x = y^2 x y^{-2} = x^{n^2}$, so $n^2 \equiv 1 \pmod 7$, so $n = \pm 1 \pmod 7$.

   If $n = 1$ then $yx = xy$, so the map $G \to \mathbb{Z}_7 \times \mathbb{Z}_2$ that sends $x^i y^j$ to (i, j) is an isomorphism.

   If $n = -1$ then $xy = yx^{-1}$ (notice that $y^{-1} = y$) so the map $G \to D_7$ that sends $x^i y^j$ to $r^i s^j$ is an isomorphism.

3. Let G be a group of order $20 = 2^2 \cdot 5$. Show that G has exactly four elements of order 5. Hint: Think about n_5.

   Solution: We have $n_5 \mid 4$, so $n_5 = 1$ or 2 or 4, and $n_5 \equiv 1 \pmod 5$, so $n_5 = 1$. Every element of order 5 generates a subgroup of order 5, which is a Sylow 5-subgroup; since there is only one such subgroup, there are four elements of order 5.

   Optional: Show if a Sylow 2-subgroup is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ then G is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$, or to $\mathbb{Z}_2 \times D_5$.

   Solution: Let x be an element of order 5, and let y and z be generators of a Sylow 2-subgroup, so $y^2 = z^2 = 1$ and $zy = yz$. Then $yxy^{-1} = x^m$ for some $m \in \mathbb{Z}$, and $m^2 \equiv 1 \pmod 5$ as we saw in lecture, so $m \equiv \pm 1 \pmod 5$. Similarly, $zxz^{-1} = x$ or x^{-1}.

   If $yxy^{-1} = x$ and $zxz^{-1} = z$ then $G \cong \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

If $yxy^{-1} = x^{-1}$ and $zxz^{-1} = x$ then the map $D_5 \times \mathbb{Z}_2 \to G$ sending $(r^i s^j, k)$ to $x^i y^j z^k$ is an isomorphism.

If $yxy^{-1} = x$ and $zxz^{-1} = x^{-1}$ then again we have $G \cong D_5 \times \mathbb{Z}_2$.

If $yxy^{-1} = x^{-1}$ and $zxz^{-1} = x^{-1}$, let $w = yz$. Then the Sylow 2-subgroup is also generated by y and w, and we have $wxw^{-1} = x$, so $G \cong D_5 \times \mathbb{Z}_2$ again.

Very optional: See what happens if a Sylow 2-subgroup is isomorphic to $\mathbb{Z}_4$.

Solution: Still let x be an element of order 5, and let y be an element of order 4. Because $\langle x \rangle$ is normal, we have $yxy^{-1} = x^n$ for some n satisfying $n^4 \equiv 1 \pmod 5$, so $n = \pm 1 \pmod 5$. If $n = 1$ then $G \cong \mathbb{Z}_5 \times \mathbb{Z}_4$. If $n = -1$ then we have another dicyclic group: the map from G to

$$\langle a, b : a^{10} = 1, b^2 = a^5, ab = ba^{-1} \rangle$$

that sends x to a^2 and y to b^{-1} is an isomorphism.

4. Let $k = \mathbb{Z}_3$, thought of as a field with elements $\{-1, 0, 1\}$. Let T be the group of 2×2 upper triangular matrices with entries in k:

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \; a, b, c \in k, a \neq 0, c \neq 0 \right\}.$$

   (a) Show that T is not Abelian, and $|T| = 12$.

   Solution: There are two choices for a, three for b, and two for c, so $|T| = 2 \cdot 3 \cdot 2 = 12$. With a view toward part (d), let

$$r = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \qquad s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   then

$$rs = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

   but

$$sr = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

   so T is not Abelian.

We know three non-Abelian groups of order 12: D_6, A_4, and

$$G = \langle a, b \mid a^6 = 1, b^2 = a^3, ab = ba^{-1} \rangle.$$

(b) Show that T has $n_3 = 1$, as follows. Consider the map $\varphi\colon T \to k^\times \times k^\times$ given by

$$\varphi \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (a, c).$$

Show that φ is a homomorphism with $|\ker \varphi| = 3$. Thus $\ker \varphi$ is a Sylow 3-subgroup and is normal.

Conclude that $T \not\cong A_4$. (What was n_3 for A_4?)

Solution: We have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}.$$

Thus φ is a homomorphism. Its kernel is the set of matrices of the form

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$

and there are three choices for b. The kernel of a homomorphism is a normal subgroup. Any two Sylow 3-subgroups are conjugate, so if one of them is normal then they are all equal, i.e. $n_3 = 1$.

Last week we saw that A_4 has $n_3 = 4$, so $T \not\cong A_4$.

(c) Show that T has a Sylow 2-subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. (Hint: Take $b = 0$.)

Conclude that $T \not\cong G$. (What were the Sylow 2-subgroups of G?)

Solution: The set of matrices with $b = 0$ is a subgroup: we have

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix} = \begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix}$$

and

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & c^{-1} \end{pmatrix},$$

so it is closed under multiplication and inverses. It is a Sylow 2-subgroup of T, since it has order $2^2 = 4$. It is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, via the map sending $(i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ to

$$\begin{pmatrix} (-1)^i & 0 \\ 0 & (-1)^j \end{pmatrix}.$$

But last week we saw that a Sylow 2-subgroup of G is isomorphic to $\mathbb{Z}_4$. Note that by Sylow's second theorem we know that any two Sylow 2-subgroups of a given group are conjugate, hence isomorphic.

(d) So we must have $T \cong D_6$. Find an element $r \in T$ with $r^6 = 1$, and an element $s \in T$ with $s^2 = 1$ and $rs = sr^{-1}$.

Solution: The matrices were given in part (a).

(e) Let T act on the set of column vectors $V = k^2$. Show that there are three orbits:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} * \\ \pm 1 \end{pmatrix}$$

Write the six elements of the last orbit in a hexagon, in such a way that your matrix r from part (d) acts as a rotation, and s acts a reflection.

Solution: Clearly the zero vector is in an orbit by itself: we have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Next, we have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \pm a \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

so the set

$$\left\{ \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \right\}$$

forms an orbit. Next, we have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d \\ \pm 1 \end{pmatrix} = \begin{pmatrix} ad \pm b \\ \pm c \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & d \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} d \\ \pm 1 \end{pmatrix},$$

so the set

$$\left\{ \begin{pmatrix} * \\ \pm 1 \end{pmatrix} \right\}$$

forms an orbit.

Finally, the hexagon is

$$
\begin{tikzcd}
 & \begin{pmatrix}0\\1\end{pmatrix} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{pmatrix}1\\-1\end{pmatrix} \arrow[d, dash] & & \begin{pmatrix}-1\\-1\end{pmatrix} \arrow[d, dash] \\
\begin{pmatrix}1\\1\end{pmatrix} \arrow[dr, dash] & & \begin{pmatrix}-1\\1\end{pmatrix} \arrow[dl, dash] \\
 & \begin{pmatrix}0\\-1\end{pmatrix}. &
\end{tikzcd}
$$

The matrix r from part (a) acts by rotating clockwise, and the matrix s acts by reflecting left-to-right.