Midterm 1

Friday, February 3, 2017

1. (a) Let G be a group. Show that if $g^2 = 1$ for every $g \in G$ then G is Abelian.
   (b) Give an example of an Abelian group G which does not satisfy $g^2 = 1$ for every $g \in G$.
   (c) Give an example of a non-Abelian group.

2. Let F be the field $\mathbb{Z}/2 = \{0, 1\}$, and let $G = \mathrm{GL}_2(F)$ be the group of 2×2 matrices with entries in F and determinant $\neq 0$.
   (a) List the elements of G. Hint: The two columns have to be different, and neither can be zero.
   (b) Let G act on the vector space $V = F^2$ in the usual way: matrix times column vector. Describe the orbit and stabilizer of the vector $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.
   (c) Use this action to show that $G \cong S_3$.

3. Continue to let $F = \mathbb{Z}/2$, and now let $G = \mathrm{GL}_3(F)$ act on $V = F^3$. We will use this action to compute the order of G.
   (a) Show that the orbit of $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ is $V \setminus \{0\}$.
   (b) How many elements does $V \setminus \{0\}$ have?
   (c) Describe the stabilizer of $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, and find its order.
   (d) Use the orbit-stabilizer theorem to find $|G|$.