Midterm 2

Friday, March 3, 2017

1. (a) Show that if $G$ is a group and $H \subset G$ is a subgroup of index 2 then $H$ is normal.
   (b) Give an example of a group $G$ and subgroup $H \subset G$ of index 3 that is not normal.
   (c) Give an example of a group $G$ and subgroup $H \subset G$ of index 3 that is normal. Describe the quotient.

2. (a) Let $G$ be a group, let $A$ be an Abelian group, and let $\varphi: G \to A$ be a homomorphism. Show that if two elements $x, y \in G$ are conjugate then $\varphi(x) = \varphi(y)$.
   (b) Let $G$ be a group, let $H \subset G$ be a normal subgroup of index 3, and let $g \in G$. Show that $g$ is conjugate to $g^{-1}$ then $g \in H$.
      Hint: Observe that $G/H \cong \mathbb{Z}/3$, and apply part (a) to the natural projection $\varphi: G \to G/H$.
   (c) Give an example of a group $G$, a subgroup $H \subset G$ of index 3 that is not normal, and an element $g \in G$ such that $g$ is conjugate to $g^{-1}$ but $g \notin H$.

3. (a) Show that $(1\ 2\ 3)$ and $(1\ 3\ 2)$ are not conjugate in $A_4$.
      Hint: Apply 2(b) with
      $$H = \{1, \ (1\ 2)(3\ 4), \ (1\ 3)(2\ 4), \ (1\ 4)(2\ 3)\}.$$ 
   (b) Show that $(1\ 2\ 3)$ and $(1\ 3\ 2)$ are conjugate in $S_4$.
   (c) Show that $(1\ 2\ 3)$ and $(1\ 3\ 2)$ are conjugate in $A_5$. 