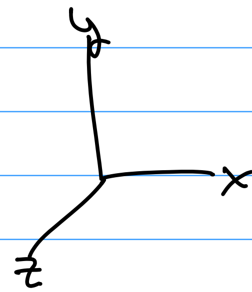
Math 607: Homological Algebra

Tor: what is it good for?

Motivation / Goals

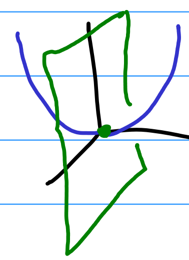
① k = field

$R = k[x,y,z]$



$I = (y - x^2, z)$

$J = (x)$

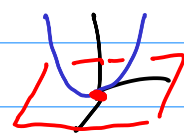


transverse

$$R/I \otimes R/J = R/(I+J) = R/(x,y,z).$$

$$\mathrm{Tor}_1^R(R/I, R/J) = 0$$

$J' = (y)$



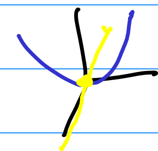
not transverse, but $\cap$ is exp. dim.

$$R/I \otimes R/J' = R/(I+J') = R/(x^2, y, z)$$

$$\neq R/(x,y,z)$$

$$\mathrm{Tor}_1^R(R/I, R/J') = 0$$

$K = (x, y)$

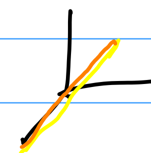


even less transverse.
codim 2 $\cap$ codim 2
= codim 3 < 4

$$R/_I \otimes R/_K = R/_{(I+K)} = R/_{(x,y,z)}$$

$$\mathrm{Tor}_1^R(R/_I, R/_K) = R/_{(x,y,z)}$$

$$\mathrm{Tor}_2 = 0$$



$$R/_K \otimes R/_K = R/_K$$

codim 2 $\cap$ codim 2 = codim 2 < 4

$$\mathrm{Tor}_1(R/_K, R/_K) = (R/_K)^2$$

$$\mathrm{Tor}_2 = R/_K$$

$$\mathrm{Tor}_3 = 0$$

In general, if $I, J \subset \mathbb{C}[x_1, \dots x_n]$

cut out $X, Y \subset \mathbb{C}^n$ smooth

Know: $\operatorname{codim}(X \cap Y) \leq \operatorname{codim} X + \operatorname{codim} Y$
difference tells the last
non-zero $\mathrm{Tor}_i(R/_I, R/_J)$.

(2) R = Noetherian comm. ring.

M = fin. gen. module

proj dim (M) = length of shortest projective resolution

glob. dim (R) = max(proj dim(modules))

Serre: glob dim $< \infty$ iff R is <u>regular</u>

if $k = \bar{k}$ and $R = k[x_1, \dots x_n]/I$

then regular $\Leftrightarrow$ smooth

but $\mathbb{Z}$ is also regular

$\mathbb{Z}[\sqrt{-5}]$ is regular

$\mathbb{Z}[\sqrt{-3}]$ is not...

$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is.