Basics

R a ring

Def. a seq. of left/right/bimodules

$$L \xrightarrow{f} M \xrightarrow{g} N$$

is *exact* if $\ker g = \operatorname{im} f$

Prop. $0 \to M \xrightarrow{g} N$ is exact iff g is injective

$L \xrightarrow{f} M \to 0$ is exact iff f is surjective.

Def. a *short exact sequence* (s.e.s.)

is $0 \to L \to M \to N \to 0$

Frank: it just means $N = M/L$!

say M is an *extension* of N by L.

Split $\quad 0 \to L \to L \oplus N \to N \to 0$

Prop $\quad 0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$ is split
(iso to the split one above)

iff $\exists$ $M \underset{s}{\overset{p}{\rightleftarrows}} N$ $\quad p \circ s = 1_N$

iff $\exists$ $L \underset{i}{\overset{r}{\rightleftarrows}} M$ $\quad r \circ i = 1_L$

sketch pf

if I have s,

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, equal] & M \arrow[r, "p"] \arrow[d, shift right, "\begin{pmatrix}1-sp\\ p\end{pmatrix}"'] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"'] & L \oplus N \arrow[r, "(0\ 1)"'] \arrow[u, shift right, "(i\ s)"'] & N \arrow[r] & 0
\end{tikzcd}$$

examples $\quad 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$

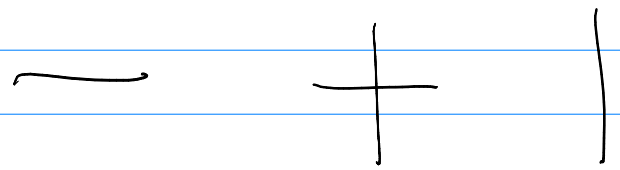
is not split.

Sim: if $R = k[x,y]$

$$0 \to R \xrightarrow{x} R \to R/x \to 0$$

or:

$$0 \to R/y \xrightarrow{x} R/xy \to R/x \to 0$$



If $0 \to L \to M \to N \to 0$ is exact,

apply $\mathrm{Hom}_R(D,-)$

$$0 \to \mathrm{Hom}(D,L) \to \mathrm{Hom}(D,M) \to \mathrm{Hom}(D,N) \cancel{\to 0}$$

$\mathrm{Hom}(D,-)$ is left exact.

$\mathrm{Ext}^*(D,-)$ will be the right derived functor.

apply $\mathrm{Hom}(-,D)$

$$0 \to \mathrm{Hom}(N,D) \to \mathrm{Hom}(M,D) \to \mathrm{Hom}(L,D) \cancel{\to 0}$$

also left exact ((contravariant $\Rightarrow$ confusing)

$\mathrm{Ext}^*(-,D)$ is the right der. functor.

apply $D \otimes -$ or $- \otimes D$

$$\cancel{0 \to} D \otimes L \to D \otimes M \to D \otimes N \to 0$$

right exact. left der. functor is $\mathrm{Tor}_*(D,-)$

Frank: difference between

$$\mathrm{Hom}(D, M)/\mathrm{Hom}(D, L) \quad \text{and} \quad \mathrm{Hom}(D, M/L)$$

lives in $\mathrm{Ext}^1(D, L)$.

difference between

$$D \otimes \ker(M \twoheadrightarrow N) \quad \text{and}$$

$$\ker(D \otimes M \twoheadrightarrow D \otimes N)$$

lives in $\mathrm{Tor}_1(D, N)$

<u>Def</u> P is <u>projective</u> if $\mathrm{Hom}(P, -)$ is exact

Q is <u>injective</u> if $\mathrm{Hom}(-, Q)$ is exact

A is <u>flat</u> if $A \otimes -$ is exact

<u>Prop</u> free $\Rightarrow$ projective $\Rightarrow$ flat

$M = R^{\oplus n}$ or maybe infinite $\oplus$

projective $\Leftrightarrow$ direct summand of free.

$\exists P'$ s.t. $P \oplus P' =$ free.

injective is more obscure,
but less important.

Next time: for fin. gen. module / Noeth. comm. ring

proj $\Leftrightarrow$ locally free $\Leftrightarrow$ flat

but we also care about

$R = k[x]$, $\quad M = k[x, 1/x]$

flat, not fin. gen, not proj.

or $R = \mathbb{Z}$ $\quad M = \mathbb{Z}[\frac{1}{2}]$ ...