Working towards:

for Noeth. comm. ring,

fin gen module is

proj. iff flat iff loc. free.

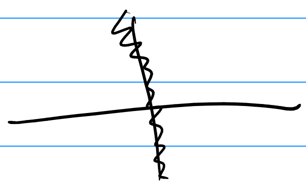
Localization

geom: an open subset
or germ of space at a point
or formal nbd.

$$k[x,y] \rightsquigarrow k[x,y,\tfrac{1}{x}]$$

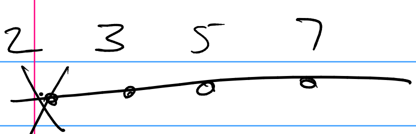
max'l ideals are $(x-a, y-b)$

same, except lose $(x, y-b)$



$$\mathbb{Z} \rightsquigarrow \mathbb{Z}[\tfrac{1}{2}]$$

max'l ideals are

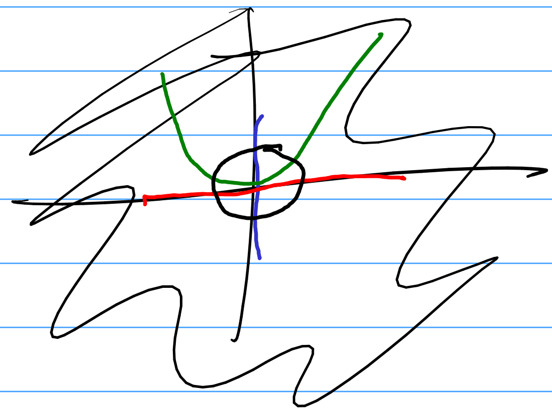


3, 5, 7, 11...

Can localize at a max'l ideal $\underline{m} \subset R$

invert everything $\underline{\text{not}}$ in $\underline{m}$.

$k[x,y]_{\underline{m}=(x,y)}$



$\shortparallel$

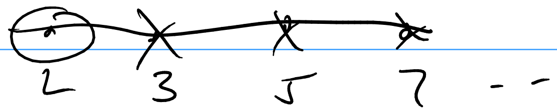
$\left\{ \frac{f}{g} \;\middle|\; \begin{array}{c} g(0,0) \neq 0 \\ g \notin \underline{m} \end{array} \right\}$

all max'l ideals are gone, except (x,y).

Still many primes: (x) (y) $(y-x^2)$

"germ"

$\mathbb{Z}_{(2)} = \left\{ \frac{a}{b} \;\middle|\; b \text{ odd.} \right\}$



Def: a comm. ring R is <u>local</u> if it has a unique max'l ideal.

could go further to <u>complete</u> local rings:

$k[[x,y]]$ and $\mathbb{Z}_2$

those are also local rings.

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could go all the way to the fraction field

$k(x,y)$ or $\mathbb{Q}$.

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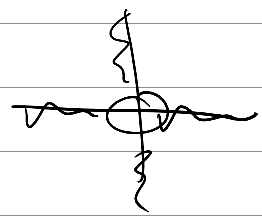
also interested in non-int. domains

$R = k[x,y]/xy$ $\quad\quad R_{(x,y)}$



Read Eisenbud §2.1 and 2.2.

localization of modules

= restricting to an open set
or a germ or formal nbd.

R = comm. ring.
$U \subset R$ mult subset
M an R-mod.

$R[U^{-1}]$.

$$M[U^{-1}] = \left\{ \frac{m}{u} \;\middle|\; m \in M,\ u \in U \right\} \Big/$$

$\frac{m}{u} = \frac{n}{v}$

if $(mv - nu)x = 0$
some $x \in U$.

check:

$$M[U^{-1}] \cong M \otimes_R R[U^{-1}].$$

also: if $0 \to L \to M$ is exact

then $0 \to L[U^{-1}] \to M[U^{-1}]$ is exact

So localizing modules is exact

$\otimes R[U^{-1}]$ is exact

$R[U^{-1}]$ is a flat R-module.

$R[u^{-1}]$ is almost never a fin gen R-module.

Next time: proj $\Leftrightarrow$ "loc proj"

flat $\Leftrightarrow$ "loc flat"

Later: over a loc ring,

fin gen module is free $\Leftrightarrow$ proj $\Leftrightarrow$ flat.