Some asked: favorite flat module?

fin. gen. proj. mod / localization

Why the name "flat"?

Fact: let $k = \bar{k}$

$I \subset k[x_1 \dots x_m]$ cut out $X \subset k^m$ smooth connected

$J \subset k[y_1 \dots y_n]$ cuts out $Y \subset k^n$ sm. conn.

$f: Y \to X$ corresp. to ring map

$R/I \to S/J$

S/J is a flat R/I-mod. iff

fibers $f^{-1}(x)$ are all same dim (or empty)

Non-example: $Y = \{y = zx\} \subset k^3$

$X = k^2$ $\quad f(x,y,z) = (x,y)$

$f^{-1}(0,0) =$ line

$f^{-1}(\text{other point}) = \text{pt or } \emptyset$.

$k[x,y,z]/_{y-zx}$ is not flat / $k[x,y]$.

Last time: localization

R = comm. ring.
$U \subset R$ mult subset
$R[U^{-1}]$.

Today: flat, proj. are local conditions.

Lemma (Eisenbud Lemma 2.8 b):

an R-module M is zero iff
$M_{\mathfrak{m}} = 0 \quad \forall$ maximal $\mathfrak{m} \subset R$.

Pf: let $x \in M$,

$I = \mathrm{Ann}(x) = \{ r \in R \mid rx = 0 \} \subset R$

if $x \neq 0$ then I is a proper ideal
$\Rightarrow I \subset$ some $\mathfrak{m} \subset R$

image of x is $M_{\mathfrak{m}}$ is not zero:

else $\frac{x}{1} = \frac{0}{1}$ i.e. $r(x \cdot 1 - 0 \cdot 1) = rx = 0$
for some $r \notin \mathfrak{m}$

$\Rightarrow \Leftarrow$

$\boxed{R}$

Prop: an R-mod A is flat iff $A_{\underline{m}}$ is flat over $R_{\underline{m}}$ $\forall$ max'l $\underline{m} \subset R$

Pf. 1^{st} suppose A is flat.

let $0 \to L \to M \to N \to 0$

be an exact seq. of R_m-mod.

$- \otimes_{R_m} A_m$ is the same as $- \otimes_R A$ (as a functor R_m-mod $\circlearrowleft$) (don't need flat here.)

$$\left(L \otimes_R A = L \otimes_{R_m} R_m \otimes_R A = L \otimes_{R_m} A_m \right)$$

so $- \otimes_{R_m} A_m$ is exact.

2^{nd}: suppose A is not flat.

let $0 \to L \to M \to N \to 0$ be an exact seq of R-mod

s.t. $\cancel{0 \to} L \otimes_R A \to M \otimes_R A \to N \otimes_R A \to 0$

$0 \to K \nearrow$

$K \neq 0$, so $\exists$ $\underline{m}$ s.t. $K_{\underline{m}} \neq 0$.

localize: $0 \to K_{\underline{m}} \to L_{\underline{m}} \otimes_{R_m} A_{\underline{m}} \to M_{\underline{m}} \otimes_{R_m} A_{\underline{m}} \to N_{\underline{m}} \otimes_{R_m} A_{\underline{m}} \to 0$

So tensoring $0 \to L_m \to M_m \to N_m \to 0$
exact seq of R_m-mod
with A_m $\rightsquigarrow$ result is not exact

$\boxtimes$

Next time: if M is fin. presented R-mod

then $\mathrm{Hom}_R(M, N) \otimes_R R_m = \mathrm{Hom}_{R_m}(M_m, N_m)$

Application: P is a proj. R-mod
iff $P_{\underline{m}}$ is a proj. $R_{\underline{m}}$-mod $\forall \underline{m}$
(assume P fin presented?)

Worksheet: same as last time.