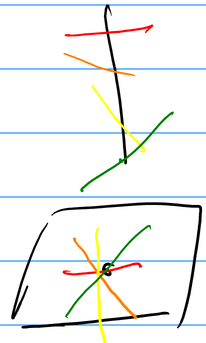
In the live lecture, my good tablet malfunctioned and afterward my computer froze and I lost the file. So this won't quite match the video.

Last time, saw that

$$R = k[x,y] \longrightarrow S/J = k[x,y,z]/_{y-zx} \quad \text{is not flat,}$$

because dim of fibers jumps.



$$\mathrm{Tor}_1^R\left(R/_{(x,y)}, S/_J\right) \neq 0$$

Agrees with what we said on 1st day:

$$\mathrm{Tor}_1^R\left(R/_{(x,y)}, S/_J\right) = \mathrm{Tor}_1^S\left(\underbrace{S/_{(x,y)}}_{\text{line}}, \underbrace{S/_J}_{\text{surface}}\right) \neq 0$$

codim 2 ∩ codim 1 = codim 2 < 3.

What about injectives modules?

- Rarely fin gen unless R is Artinian.
- Not geometrically significant.

Fact that we used last time:
if a functor F preserves short exact seqs then it preserves all exact seqs.

Given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ exact,

split it into short exact seqs,

$$0 \to \ker f \to A \to \operatorname{im} f \to 0$$
$$0 \to \ker g \to B \to \operatorname{im} g \to 0$$
$$0 \to \ker h \to C \to \operatorname{im} h \to 0$$
$$0 \to \operatorname{im} h \to D \to \operatorname{coker} h \to 0$$

$\ker g = \operatorname{im} f \qquad \ker h = \operatorname{im} g = \operatorname{coker} f$

apply F and string them back together:

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & \\
 & & & F\operatorname{im} f \arrow[ur] \arrow[dr] & & & & F\operatorname{im} h \arrow[ur] \arrow[dr] & & & \\
 & & FA \arrow[ur] \arrow[rr] & & FB \arrow[rr] \arrow[dr] & & FC \arrow[rr] \arrow[ur] & & FD \arrow[dr] & & \\
 & F\ker f \arrow[ur] & & & & F\operatorname{im} g \arrow[ur] \arrow[dr] & & & & F\operatorname{coker} h \arrow[dr] & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

exact!

=

Eisenbud Prop 2.10:

Let R be a comm. ring, M a finitely presented R-module, and $R \to S$ a ring hom s.t. S is flat $/R$.

(In our application, $S = R[u^{-1}]$.)

Then the natural map

$$\mathrm{Hom}_R(M, N) \otimes S \to \mathrm{Hom}_S(M \otimes S, N \otimes S)$$

is an iso.

Sketch: pick a presentation $\underbrace{R^m}_{\text{relations}} \to \underbrace{R^n}_{\text{generators}} \to M \to 0$

apply $\mathrm{Hom}_R(-, N)$ and then $- \otimes S$ which is exact

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M, N) \otimes S \arrow[r] \arrow[d] & N^n \otimes S \arrow[r] \arrow[d, "\cong"] & N^m \otimes S \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_S(M \otimes S, N \otimes S) \arrow[r] & (N \otimes S)^n \arrow[r] & (N \otimes S)^m
\end{tikzcd}$$

Or $- \otimes S$ to get $S^m \to S^n \to M \otimes S \to 0$
and then $\mathrm{Hom}_S(-, N \otimes S)$

Check that the natural maps commute w/ everything, and the last two are isos.

Then the first one is an iso by the 5-lemma.

$\square$

Non-example when M is not f.g. gen:

$R = k[x]$ $\quad M = k[x, \frac{1}{x}]$, $\quad N = R$

$\mathrm{Hom}_R(M, N) = 0$

but localize at $\underline{m} = (x-1)$...

$M_{\underline{m}} = R_{\underline{m}} = N_{\underline{m}}$

$\mathrm{Hom}_{R_{\underline{m}}}(M_{\underline{m}}, N_{\underline{m}}) = R_{\underline{m}}$

Similar: $R = \mathbb{Z}$, $\quad M = \mathbb{Z}[\frac{1}{2}]$, $\quad N = \mathbb{Z}$

$\mathrm{Hom}(M, N) = 0$,

but localize at $\underline{m} = (3)$ and

$M_{\underline{m}} = R_{\underline{m}} = N_{\underline{m}}$.