One asked: is <u>exact</u> equiv to <u>preserves inj. and surj.</u>?

No: let $R = k$ so $R\text{-mod} = k\text{-vec}$

Functor: Λ^n (exterior power)

if $0 \to U \to V$ (fin dim'l) then $0 \to \Lambda^n U \to \Lambda^n V$

if $V \to W \to 0$ then $\Lambda^n V \to \Lambda^n W \to 0$

but if $0 \to U \to V \to W \to 0$ then $0 \to \Lambda^n U \to \Lambda^n V \to \Lambda^n W \to 0$ is almost never exact.

$$0 \to \mathbb{C}^2 \to \mathbb{C}^4 \to \mathbb{C}^2 \to 0$$

apply Λ^2

$$0 \to \mathbb{C}^1 \to \mathbb{C}^6 \to \mathbb{C}^1 \to 0$$

can't be exact.

Further Q: does <u>additive</u> save us?

So far this week: R comm. Noetherian
M a fin. gen module.

M flat $\longleftrightarrow$ $M_{\underline{m}}$ flat / $R_{\underline{m}}$ $\forall$ max'l $\underline{m} \subset R$

M proj $\longleftrightarrow$ $M_{\underline{m}}$ proj / $R_{\underline{m}}$ $\forall$ - - -

+ many examples + digressions.

Today: R Noetherian local ring

= comm w/ unique max'l $\underline{m} \subset R$

$k := R/\underline{m}$ residue field

fav examples:

$\mathbb{Z}_{(2)} = \left\{ \frac{a}{b} \,\middle|\, b \text{ odd} \right\}$ $\quad \underline{m} = (2)$ $\quad k = \mathbb{Z}/2$

$R = k[x,y]_{(x,y)} = \left\{ \frac{f}{g} \,\middle|\, g(0,0) \neq 0 \right\}$ $\quad \underline{m} = (x,y)$ $\quad k = k$

one more:

R = power series in x,y coeffs in $\mathbb{C}$
with >0 radius of conv.

(If we said radius of conv $>$ fixed r,
not local: max'l ideal $\forall$ point in open disc
of radius r. But if we don't bound radius
below...)

Nakayama's lemma: R, $\underline{m}$, k as above
M a fin. gen. module.

then $M = 0$ iff $M \otimes k = 0$.

<u>Proof</u>: notice: $M \otimes k = M / \underline{m} M$.

(why? take $0 \to \underline{m} \to R \to k \to 0$
and $\otimes_R M$.)

so $M \otimes k = 0$ iff $M = \underline{m} M$.

let $u_1 \dots u_r \in M$ be generators.

each $u_i \in \underline{m} M$, so write

$$u_i = \sum a_{ij} u_j \qquad a_{ij} \in \underline{m}$$

Consider

$$A = \begin{pmatrix} a_{11} - 1 & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} - 1 & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} - 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

then $A \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

is A invertible? then we win.
$u_i = 0$ so $M = 0$.

$A \equiv -I \mod \underline{m}$

$\det A \equiv \pm 1 \mod \underline{m}$

so $\det A \notin \underline{m}$

so $\det A$ is a unit (bec. R is local!)

so A is invertible. $\square$

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Counterexample when M is <u>not</u> fin gen.

$R = \mathbb{Z}_{(2)}$ $M = \mathbb{Q}$ $M \otimes k = \mathbb{Q} \otimes \mathbb{Z}/2 = 0$

Similar: $R = k[x,y]_{(x,y)}$ $M = k(x,y)$ field of rational functions

$k = R/(x,y)$ $M \otimes k = 0$.

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Worksheet: if N is fin gen

then $f: M \to N$ is surj.

iff $f \otimes 1 : M \otimes k \to N \otimes k$ is surj.

(doesn't work with injective.)

Corollary: let M be fin. gen
then minimal # of generators
$= \dim_k (M \otimes k)$.

Proof: choose basis $v_1 \dots v_r$ for $M \otimes_R k = M/\mathfrak{m}M$.

lift these to $u_1, \dots, u_r \in M$.

claim they generate M.

$$R^r \xrightarrow[(u_1 \dots u_r)]{} M \to 0 \text{ is surj.}$$

$$\text{iff} \quad k^r \xrightarrow[(v_1 \dots v_r)]{} M \otimes k \to 0 \text{ is surj.}$$

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Thm R Noeth. local ring

M a fin gen module.

free $\iff$ proj $\iff$ flat.

Pf in general, free $\Rightarrow$ proj. $\Rightarrow$ flat,

so suppose M is flat.

choose minimal generators

$$R^r \to M \to 0$$

$$0 \to \ker \to R^r \to M \to 0$$

$\otimes k$:

$$\underbrace{\mathrm{Tor}_1^R(M,k)}_{} \to \ker \otimes k \to k^r \xrightarrow{\cong} M \otimes k \to 0$$

zero because M is flat.

So $\ker \otimes k = 0$.

$\ker$ is fin. gen. because R is Noeth.

so $\ker = 0$

so $M \cong R^r$. $\blacksquare$

Next week: Ext and Tor