Some asked:

$k=\bar{k}$

$R=k[x_1 \dots x_n]$

$I=(f_1, \dots, f_k)\subset R$ radical

cuts out $X=\{\vec{x} \mid f_1(\vec{x})=\dots=f_k(\vec{x})=0\}\subset k^n$

What does it mean for X to be smooth?

if $k=\mathbb{C}$, means X is a mfld.

yes: $x^2+y^2=z^2+1$ no: $x^2+y^2=z^2$

not smooth at 0.

equiv. to a statement about derivatives

so take that as the def when $k\neq\mathbb{C}$.

last Friday: Nakayama's lemma:

if R = local ring
w/ max'l ideal $\underline{m}$
and res. field $k = R/\underline{m}$

then a fin. gen. module M is 0
iff $M \otimes k = 0$.

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today: "fibers" of modules.

keep setup above:
$k = \bar{k}$
$R = k[x_1 \cdots x_n] /$ radical ideal

max'l ideals $\underline{m} \subset R \longleftrightarrow$ points in $X \subset k^n$

(really just need a Noeth. comm. R
with $\cap$ max'l ideals $= 0$

could do $\mathbb{Z}$ or $\mathbb{Z}[\sqrt{-3}]$ or $\mathbb{Z}[\frac{1}{2}]$
or poly rings over them...

__not__ a local ring of dim > 0
__not__ $k[t]/t^2$.)

let M be a fin. gen R-mod

$\forall$ max'l $\underline{m} \subset R$ (= point of X)

get $R \to R/\underline{m} \cong k$

last week: studied $M_{\underline{m}} = M \otimes_R R_{\underline{m}}$

now consider $M/\underline{m}M = M \otimes_R R/\underline{m} = M \otimes_R k$

a fin-dim'l k-vector space.

last time: dim = min # of gens of $M_{\underline{m}}$ over $R_{\underline{m}}$

$\dim_k (M \otimes_R k)$ varies from point to point.

how does it look?

choose a presentation $R^m \xrightarrow{A} R^n \to M \to 0$

A is an $n \times m$ matrix w/ entries in R.

$k^m \xrightarrow{A \otimes k} k^n \to M \otimes k \to 0$

$\dim_k (M \otimes k) = n - \operatorname{rank}(\underbrace{A \otimes k}_{\text{entries in } k.})$

for each integer d consider the radical of the ideal gen'd by $(n-d+1)\times(n-d+1)$ minors of A

cuts out a set $X_d \subset X \subset k^n$ on which fibers of M have $\dim \geq d$.

Example

$R = k[x,y,z,u,v,w]$ $\qquad X = k^6$

$$A = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & u & v \\ -y & -u & 0 & w \\ -z & -v & -w & 0 \end{pmatrix} \quad \text{skew}$$

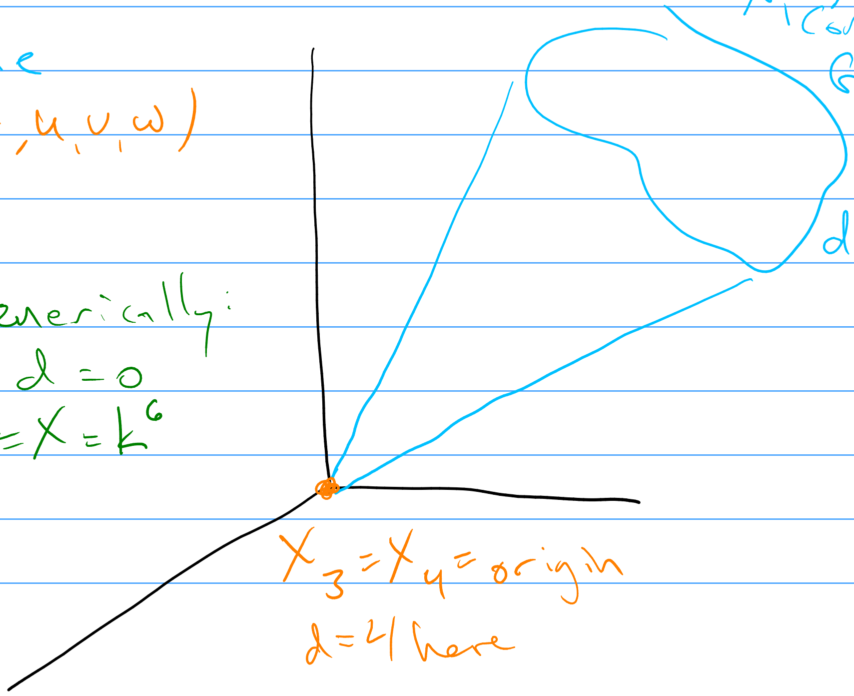
$M = \operatorname{coker}(A : R^4 \to R^4)$

$\sqrt{\text{ideal gen. by } 4\times 4 \text{ minors}}$: $(\underline{xw - yv + zu})$

3×3: same

2×2: (x,y,z,u,v,w)

1×1: same



recall $d = \dim_k(M \otimes R/\mathfrak{m})$

X_d's are called the "flattening stratification" of M.

in this example,

on $k^6 \setminus$ cone fiber of M are 0-dim'l.

on cone $\setminus$ origin — — — 2-dim'l

at origin — — — 4-dim'l.

Thm let R be a comm. ring w/ $\bigcap$ max'l ideals $=0$

let M be a fin. pres. module.

if $\exists\, d \in \mathbb{Z}_{\geq 0}$ s.t. $\forall$ max'l $\underline{m} \subset R$,

$M \otimes R/\underline{m}$ has dim $= d$ as an $R/\underline{m}$-v.s.

then M is proj / flat / loc. free.

(Eisenbud Ex. 20.13)

Pf. choose a pres. $R^m \xrightarrow{A} R^n \to M \to 0$

let $r = n-d$

know: every $(r+1)\times(r+1)$ minor of $A \otimes R/\underline{m}$ vanishes $\forall\, \underline{m}$.

so minor is in every $\underline{m}$

so actually $=0$.

fix $\underline{m}$, localize there. want: $M_{\underline{m}} \cong R_{\underline{m}}^{d}$.

Some $r\times r$ minor of $A \otimes R/\underline{m}$ doesn't van,

so doesn't van in $R_{\underline{m}}$.

for simplicity, assume upper left minor

$$R_{\underline{m}}^{m} \xrightarrow{A_{\underline{m}}} R_{\underline{m}}^{n} \longrightarrow M_{\underline{m}} \longrightarrow 0$$

$$\begin{array}{c c} & \begin{array}{cc} r & m-r \end{array} \\ \begin{array}{c} r \\ n-r=d \end{array} & \left(\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right) \end{array} \begin{pmatrix} 1 & -B^{-1}C \\ 0 & 1 \end{pmatrix}$$

do row + col operations to make $A_{\underline{m}}$ into

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\ \hline & \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} \end{array}\right)$$

those are autos of $R_{\underline{m}}^{m}$ and $R_{\underline{m}}^{n}$

$\rightsquigarrow$ doesn't change coker.

coker of $\left(\begin{array}{c|c} \begin{matrix} 1 & \\ & 1 \end{matrix} & \\ \hline & \begin{matrix} 0 & \\ & 0 \end{matrix} \end{array}\right)$ is $R_{\underline{m}}^{d}$.

$\boxed{\Sigma}$