Last time: fibers of modules

flattening stratification. many confused.

R = Noeth. comm. ring

M = fin gen module. consider

$$\{\text{max'l } \underline{m} \subset R\} \xrightarrow{\varphi} \mathbb{Z}_{\geq 0}$$

$$\underline{m} \longmapsto \dim_{R/\underline{m}} (M \otimes R/\underline{m})$$

if $R = \mathbb{C}[x_1 \cdots x_n]/\text{radical ideal}$

then max'l $\underline{m} \subset R \longleftrightarrow$ points of $X \subset \mathbb{C}^n$

has a topology

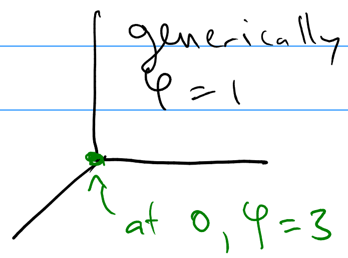
φ is upper semi-cont: in the limit, can only jump up.

given a pres $R^m \xrightarrow{A} R^n \to M \to 0$

$\varphi^{-1}([d,\infty))$ = set of points where $\dim M \otimes R/\underline{m} \geq d$

cut out by $(n-d+1) \times (n-d+1)$ minors of A

worksheet: studied $\mathbb{C}[x,y,z]$ and $A = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix}$



$\mathbb{C}^3$

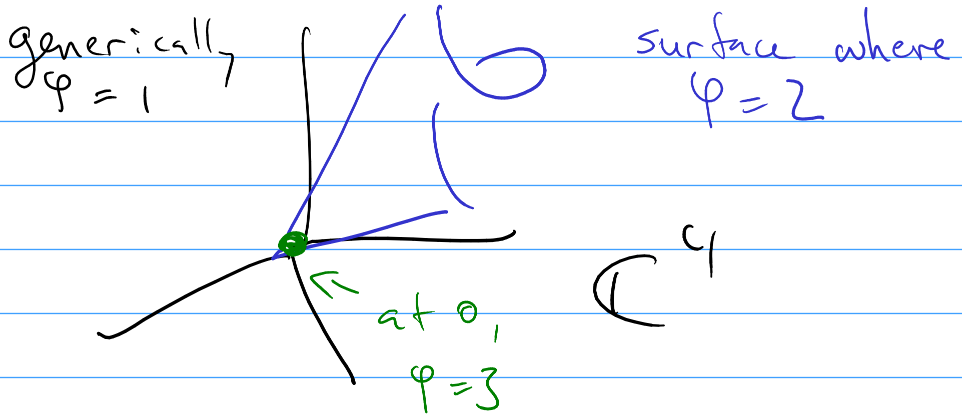
But! $M = \operatorname{coker} A \neq R \oplus (R/(x,y,z))^2$

which would have same φ,

$\operatorname{Hom}(M,M) = R$

also studied $R = (x, y, z, w)$

$M = (xz - y^2, xw - yz, yw - z^2)$



Thm: if $\cap \underline{m} = 0$ and φ is const then M is proj.

Converse if no idempotents / Spec R is connected

Example 4: $R = \mathbb{Z}[\sqrt{-5}]$

$M = (2, 1+\sqrt{-5})$ seen that it's proj.

(not $R/I = \mathbb{Z}/2$)

fun to check: M = coker of

$$R^2 \xrightarrow{A = \begin{pmatrix} 2 & 1+\sqrt{-5} \\ 1-\sqrt{-5} & 3 \end{pmatrix}} R^2$$

2×2 minors of A: $2\cdot 3 - (1+\sqrt{-5})(1-\sqrt{-5}) = 6 - 6 = 0$.

1×1 minors: $(2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}) = (1)$

φ is constantly 1.

Some asked: "fibers" of what map?

$$R \longrightarrow \mathrm{Sym}^* M = R \oplus M \oplus \underbrace{\mathrm{Sym}^2 M}_{M \otimes M / m_1 \otimes m_2 - m_2 \otimes m_1} \oplus \mathrm{Sym}^3 M \oplus \cdots$$

$$\mathrm{Spec}\, R \underset{s}{\overset{p}{\rightleftarrows}} \mathrm{Spec}\, \mathrm{Sym}^* M$$

sections s of p = dual module $\mathrm{Hom}(M, R)$

so if $M = N^*$ then

fibers of M "are" fibers of $\mathrm{Spec}\,\mathrm{Sym}^* N \to \mathrm{Spec}\, R$.

Toward Ext and Tor.

$\boxed{R = \text{general ring w/ 1, not comm or Noeth.}}$

Def. a <u>proj. resolution</u> of an R-module M is an exact seq.

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \; (\to M \to 0)$$

s.t. $\mathrm{coker}\, d_1 \cong M$.

and all P_i are projective.

<u>always</u> label d_i by its source.

distinguish P_1, P'_1, d_1, d' if you want.

injective resolutions: later.

flat resolutions: also interesting.

Examples: $R = k[x,y,z]$

res of R/x: $\quad 0 \to \underset{1}{R} \xrightarrow{x} \underset{0}{R}$ $\quad$ proj dim ≤ 1

res of $R/(x,y)$ $\quad 0 \to \underset{2}{R} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \underset{1}{R^2} \xrightarrow{(x\ y)} \underset{0}{R}$

on worksheet. $\quad$ proj dim ≤ 2

"Koszul resolution"

res of $R/(y-x^2, z)$ $\quad 0 \to R \xrightarrow{\begin{pmatrix} z \\ x-y^2 \end{pmatrix}} R^2 \xrightarrow{(y-x^2\ \ z)} R$

$S = k[x,y,z,w]/_{xz+yw}$

res of $S/(x,y)$ $\quad$ proj dim $\le \infty$ (?)

$$\cdots \to S^2 \xrightarrow{B} S^2 \xrightarrow{A} S^2 \xrightarrow{B = \begin{pmatrix} w & -z \\ x & y \end{pmatrix}} S^2 \xrightarrow{A = \begin{pmatrix} y & z \\ -x & w \end{pmatrix}} S^2 \xrightarrow{(x\ y)} S$$

Def. The proj. dimension of M is the min. length of a proj. res.

Could be ∞. examples above.

Next time: define Ext + Tor

claim that they're well-defined

use to see that those $\le$ above are $=$.