Last time: proj. res. of a module M.
proj. dim = shortest proj. res. (count from 0)
proj. dim $M = 0 \longleftrightarrow M$ is projective.

You read about Ext and Tor:
given M, N, get proj. res. of M

$$\cdots \to P_2 \to P_1 \to P_0 \; (\to M \to 0)$$

apply $\mathrm{Hom}(-, N)$, get a complex

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \mathrm{Hom}(P_2, N) \to \cdots$$

have $d^2 = 0$, but maybe not exact
define $\mathrm{Ext}^i_R(M, N) = \ker / \mathrm{im}$.
$\mathrm{Ext}^0(M, N) = \mathrm{Hom}(M, N)$

apply $- \otimes N$, get a complex

Not $M \otimes N \to 0$ here.

$$\cdots \to P_2 \otimes N \to P_1 \otimes N \to P_0 \otimes N \to 0$$

define $\mathrm{Tor}^R_i(M, N) = \ker / \mathrm{im}$.
$\mathrm{Tor}_0(M, N) = M \otimes N$.

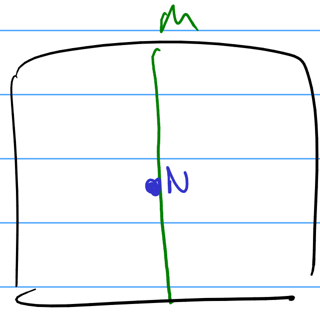
Soon: answer is indep. of proj. res.

also: $\mathrm{Tor}^i(M, N) = \mathrm{Tor}^i(N, M)$
so you could have resolved N.

Examples: $R = k[x,y]$

$M = R/x$ $\quad R \xleftarrow{x} R \leftarrow 0$ is a proj. res.

$N = R/(x,y)$ $\quad R \xleftarrow{(x\ y)} R^2 \xleftarrow{\binom{y}{-x}} R \leftarrow 0$



Compute $\mathrm{Tor}_*(M,N)$

apply $-\otimes N$ to 1st complex.

$$0 \to N \xrightarrow[=0]{x} N \to 0$$

$\mathrm{Tor}_1(M,N) = N$ $\quad \mathrm{Tor}_0(M,N) = N$

or apply $-\otimes M$ to 2nd complex

$$0 \to M \xrightarrow{\binom{y}{0}} M^2 \xrightarrow{(0\ y)} M \to 0$$

$M \xrightarrow{y} M$ inj.
so $\mathrm{Tor}_2(N,M) = 0$

why is ker/im $= N$? think a bit...

$\mathrm{Tor}_0 = M/yM = N$

learn: can resolve either, but one is easier.

also: proj. dim M, proj dim $N \geq 1$

choose shorter res $\rightsquigarrow$ Tor_1 would vanish
but it doesn't.

$\mathrm{Tor}_*(N,N)$?

apply $-\otimes N$ to 2nd res above

$$0 \to N \xrightarrow{\binom{0}{0}} N^2 \xrightarrow{(0\ 0)} N \to 0$$

$\mathrm{Tor}_0 = N \quad \mathrm{Tor}_1 = N^2 \quad \mathrm{Tor}_2 = N$

so proj. dim $N \geq 2$.

Worksheet: do Exts

From last worksheet, $R = k(x,y,z,w)$

$S = k[x,y,z,w]/xz+yw \qquad M = S/(x,y) = k[z,w]$

$$S \xleftarrow{(x\ y)} S^2 \xleftarrow{\begin{pmatrix} y & z \\ -x & w \end{pmatrix}} S^2 \xleftarrow{\begin{pmatrix} w & -z \\ x & y \end{pmatrix}} S^2 \leftarrow \text{2-periodic.}$$

$\quad ⓪ \qquad ① \qquad ②$

exact at ①? let $\binom{f}{g} \in S^2$

suppose $xf + yg = 0$ in S. lift to $\binom{\tilde f}{\tilde g} \in R^2$

then $x\tilde f + y\tilde g = (xz+yw)h$ in R

so $x(\tilde f - zh) + y(\tilde g - wh) = 0$ in R

so $\exists\, k \in R$ s.t. $\tilde f - zh = yk$ and $\tilde g - wh = -xk$

so $\begin{pmatrix}\tilde f\\ \tilde g\end{pmatrix} = \begin{pmatrix} y & z\\ -x & w\end{pmatrix}\begin{pmatrix}k\\ h\end{pmatrix}$ $\square$

exact at ②? let $A = \begin{pmatrix} y & z\\ -x & w\end{pmatrix} \in M_2(R)$

$B = \begin{pmatrix} w & -z\\ x & y\end{pmatrix} \in M_2(R)$

observe that $AB = BA = (xz+yw)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

"matrix factorization" of $xz+yw$

let $v \in S^2$ suppose $\bar{A} v = 0$
lift to $\tilde v \in R^2$ so $A\tilde v = (xz+yw)u$ $\exists u \in R^2$

so $\underbrace{BA}_{\cancel{(xz+yw)}}\tilde v = \cancel{(xz+yw)}\, Bu$

so $\tilde v = Bu$ so $v = \bar{B}\bar{u}$ $\bar u \in S^2$ $\square$

similar after that.

Compute $\mathrm{Tor}_*(M,k)$ where $k = S/(x,y,z,w)$

$$0 \leftarrow k \xleftarrow{(0\ 0)} k^2 \xleftarrow{\begin{pmatrix}0&0\\0&0\end{pmatrix}} k^2 \xleftarrow{\begin{pmatrix}0&0\\0&0\end{pmatrix}} k^2 \xleftarrow{\begin{pmatrix}0&0\\0&0\end{pmatrix}} \cdots$$

so $\mathrm{Tor}_0(M,k) = k$ $\mathrm{Tor}_i(M,k) = k^2$ $\forall i \geq 1$

so proj. dim $M = \infty$ proj. dim $k = \infty$.

$xz+yw=0$ in k^4