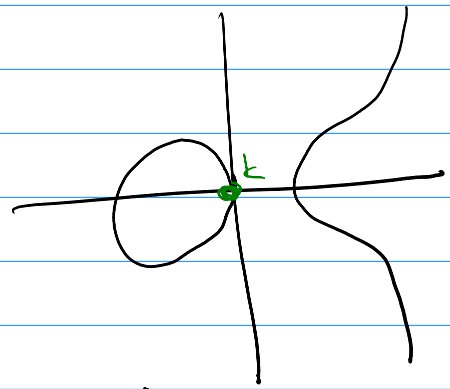
On Worksheet 3, had

$$R = k[x,y]\Big/(x^3-x-y^2)$$



look at $k = R/(x,y)$

$$x^3 - x - y^2 = x(\overbrace{x^2-1}^{z}) + y(\overbrace{-y}^{w})$$

relate to Friday's discussion of $xz+yw$

free res:

$$0 \leftarrow k \leftarrow \boxed{R} \xleftarrow{(x\ y)} R^2 \xleftarrow{\begin{pmatrix} y & x^2-1 \\ -x & -y \end{pmatrix}} R^2 \xleftarrow{\begin{pmatrix} -y & 1-x^2 \\ x & y \end{pmatrix}} R^2 \leftarrow \text{repeat}$$

still exact — adapt argument from Friday.

looks like proj. $\dim_R(k)$ might be ∞?

compute $\operatorname{Tor}_*^R(k,k)$:

take that free res. and $\otimes\, k = R/(x,y)$

$$k \xleftarrow{(0\ 0)} k^2 \underset{\text{rank}=1}{\xleftarrow{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}}} k^2 \underset{\text{rank}=1}{\xleftarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}} k^2 \underset{\text{rank}=1}{\xleftarrow{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}}} \text{repeats.}$$

$\operatorname{Tor}_0 = k$

$\operatorname{Tor}_1 = k^2/k^1 = k^1$

$\operatorname{Tor}_2 = k^1/k^1 = 0$

$\operatorname{Tor}_3 = k^1/k^1 = 0$

etc.

Maybe $\text{proj. dim}_R(k) = 1$, not ∞?

Truncate res. above

$$0 \leftarrow k \leftarrow R \leftarrow \text{ideal}(x,y) \hookleftarrow 0$$

$\hookrightarrow$ saw that this was loc. free.

Similar: $R = \mathbb{Z}[\sqrt{-5}]$, $I = (2, 1+\sqrt{-5})$ is loc. free

$$\begin{array}{ccccc} 0 \leftarrow R/I \leftarrow & R \xleftarrow{(2 \;\; 1+\sqrt{-5})} R^2 \xleftarrow{\begin{pmatrix} 1+\sqrt{-5} & 3 \\ -2 & 1-\sqrt{-5} \end{pmatrix}} R^2 \leftarrow \text{etc} \\ & \nwarrow \\ & I \leftarrow 0 \end{array}$$

Was it a coincidence that computing $\text{Tor}_*^R(k,k)$ told us $\text{proj. dim}_R(k)$?

No: can detect $\text{proj. dim}(M)$ using $\text{Tor}_*(M, R/\underline{m})$ as $\underline{m} \subset R$ max'l ideal varies.

Now: Ext and Tor are well-defined.

Recall: a (chain) complex of R-modules is

$$\cdots \longrightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$

$d^2=0$. $\quad \operatorname{im} d_{i+1} \subset \ker d_i$

homology: $H_i(M_\cdot) = \ker d_i / \operatorname{im} d_{i+1}$

a chain map $f_\cdot : M_\cdot \longrightarrow N_\cdot$

$$\begin{tikzcd}
\cdots \arrow[r] & M_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & M_i \arrow[r, "d_i"] \arrow[d, "f_i"] \arrow[dl, "s_i"'] & M_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] \arrow[dl, "s_{i-1}"'] & \cdots \\
\cdots \arrow[r] & N_{i+1} \arrow[r, "d'_{i+1}"] & N_i \arrow[r, "d'_i"] & N_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

$fd = df$

f_i takes $\ker d_i$ into $\ker d_i'$,
$\operatorname{im} d_{i+1}$ into $\operatorname{im} d'_{i+1}$,

so induces a map $H_i(M_\cdot) \longrightarrow H_i(N_\cdot)$

two chain maps $f_\cdot, g_\cdot$ are <u>homotopic</u>

if $\exists\, s_i : M_i \longrightarrow N_{i+1}$ s.t. $f-g = ds+sd$

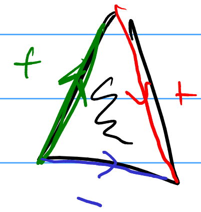
(note that s does not commute with d.)

then $f_\cdot$ and $g_\cdot$ induce the same map
$H_i(M_\cdot) \longrightarrow H_i(N_\cdot)$

Favorite example: $X =$ top. space

$$C_i(X;R) = R^{\oplus\{\text{cont maps } \sigma:\, \Delta^i \to X\}}$$

$d: C_i \to C_{i-1}$ takes the boundary



given $f: X \to Y$ cont, get a chain map

$$\begin{aligned} f_\#: C_i(X;R) &\longrightarrow C_i(Y;R) \\ \sigma: \Delta^i \to X &\longmapsto f\circ\sigma: \Delta^i \to Y \end{aligned}$$

given $f,g: X \to Y$ and a homotopy $h: X \times I \to Y$, build a chain htpy $s: C_i(X;R) \to C_{i+1}(Y,R)$ between $f_\#$ and $g_\#$.

Also recall the univ. prop. of proj. modules:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] & \\ M \arrow[r, two heads] & N \arrow[r] & 0 \end{tikzcd}$$

$\mathrm{Tor}_i^R(M, D)$ is functorial in M or D

$M \downarrow$ work

$D \hookrightarrow$ easy

given $f: M \to N$, want $f_*: \mathrm{Tor}_i^R(M, D) \to \mathrm{Tor}_i^R(N, D)$

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dashed, "\exists f_2"] \arrow[dr, "f_1 d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, dashed, "\exists f_1"] \arrow[dr, "f_0 d_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "\exists f_0"] \arrow[dr, "f\varepsilon"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & P_2' \arrow[r, "d_2'"'] & P_1' \arrow[r, "d_1'"'] & P_0' \arrow[r, "\varepsilon'"] & N \arrow[r] & 0
\end{tikzcd}$$

observe that $\varepsilon' f_0 d_1 = f \varepsilon d_1 = 0$,

so $f_0 d_1$ takes vals in $\ker \varepsilon' = \operatorname{im} d_1'$

and $P_1' \xrightarrow{d_1'} \operatorname{im} d_1' \subset P_0'$

this induces a map on Tor:

apply $- \otimes D$

still have two chain complexes

and a chain map

we get induced map on $H_i = \mathrm{Tor}_i(-, D)$

Claim: given two lifts f_i, f_i' of f,
they induce the same map on Tor_i

reduce to: $f_i - f_i'$, which lifts $0 : M \to N$,
induces 0 on Tor_i

Worksheet: produce $s_i : P_i \to P'_{i+1}$
with $ds + sd = f_i - f_i'$.

then we win.