

Showing that Ext and Tor are well-defined.

Last time: given $f: M \rightarrow N$,

$$\begin{array}{ccccccc} \dots & \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & P_0 \rightarrow M \rightarrow 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & \downarrow f \\ - & \rightarrow & P'_2 & \rightarrow & P'_1 & \rightarrow & P'_0 \rightarrow N \rightarrow 0 \end{array}$$

apply $- \otimes D$ still have chain map.
take homology, get

$$f_*: \text{Tor}_i^R(M, D) \rightarrow \text{Tor}_i^R(N, D)$$

choose different lifts $f'_i: P_i \rightarrow P'_i$

\hookrightarrow they're homotopic $f_i - f'_i = d_i s + s d_{i+1}$ $\exists s$

\hookrightarrow apply $- \otimes D$ still homotopic

\hookrightarrow induce the same map on homology.

Prop: two proj. resolutions $P_i \rightarrow M$
and $P'_i \rightarrow M$

give the same $\text{Tor}_i^R(M, \rightarrow)$.

Proof:

$$\begin{array}{ccccccc} \cdots & \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & P_0 \rightarrow M \rightarrow 0 \\ & & f_2 \downarrow & & f_1 \downarrow & & f_0 \downarrow & \downarrow 1 \\ \cdots & \rightarrow & P'_2 & \rightarrow & P'_1 & \rightarrow & P'_0 \rightarrow M \rightarrow 0 \\ & & g_2 \downarrow & & g_1 \downarrow & & g_0 \downarrow & \downarrow 1 \\ \cdots & \rightarrow & P_2 & \rightarrow & P_1 & \rightarrow & P_0 \rightarrow M \rightarrow 0 \end{array}$$

apply $- \otimes D$, get maps on homology

$$\text{Tor}_i^R(M, D) \xrightarrow{\bar{f}_i} \text{Tor}_i^R(M, D)' \xrightarrow{\bar{g}_i} \text{Tor}_i(M, D)$$

claim that $\bar{g}_i \circ \bar{f}_i = 1$

true bec. $g_i \circ f_i$ lifts $1: M \rightarrow M$

$1: P_i \rightarrow P_i$ also lifts $1: M \rightarrow M$

so $g_i \circ f_i$ and 1 induce the same map on homology. \square

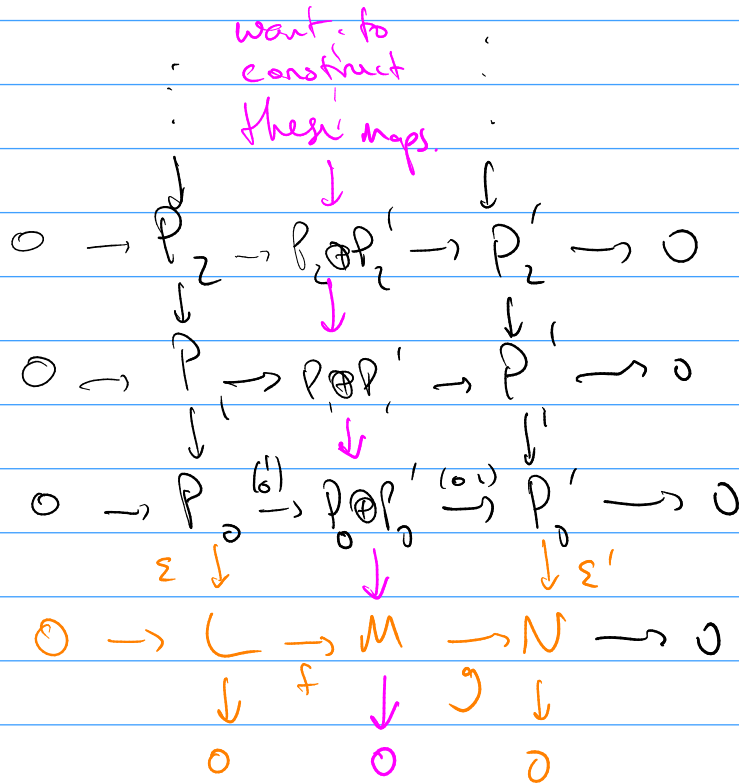
Soon: can resolve either M or N

and get the same $\text{Tor}_i^R(M, N)$.

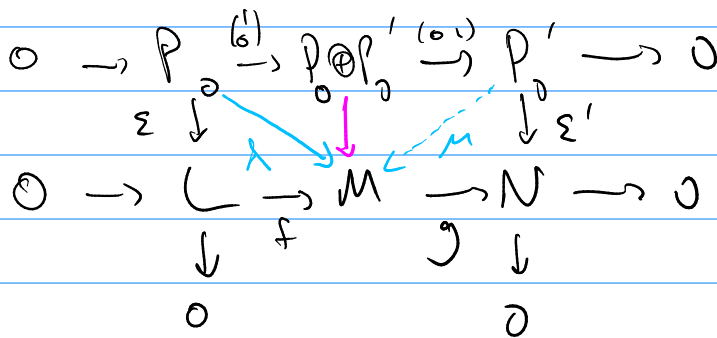
Prop given a s.e.s. $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$,

\exists simultaneous proj. resolutions.

Pf



DEF 17.1
Prop 7
+ Problem 5



$$\lambda = f \circ \varepsilon$$

$$g \circ \mu = \varepsilon'$$

put $(\lambda \mu)$ there. why does it commute?
why is it surjective?

$$(\lambda \mu) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda = f \varepsilon \quad \checkmark$$

$$g \circ (\lambda \mu) = (g \circ \lambda \quad g \circ \mu) = \begin{pmatrix} g \circ f \circ \varepsilon & \varepsilon' \end{pmatrix} = \varepsilon' \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Surj: by Snake lemma:

$$0 \rightarrow \ker \varepsilon \rightarrow \ker \rightarrow \ker \varepsilon' \rightarrow 0$$

$$\begin{array}{ccccccc}
 0 & \rightarrow & P_0 & \xrightarrow{(\iota)} & P_0 \oplus P_0' & \xrightarrow{(\iota')} & P_0' \rightarrow 0 \\
 & & \downarrow \varepsilon & & \downarrow & & \downarrow \varepsilon' \\
 0 & \rightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \rightarrow & \ker & \rightarrow & 0 \rightarrow 0
 \end{array}$$

so this is zero

and lets us continue:

$$\begin{array}{ccccccc}
 0 & \rightarrow & P_1 & \xrightarrow{(\iota)} & P_1 \oplus P_1' & \xrightarrow{(\iota')} & P_1' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \ker \varepsilon & \rightarrow & \ker(\downarrow) & \rightarrow & \ker \varepsilon' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

repeat the same argument. □

Q: is the map $P_1 \oplus P_1' \rightarrow P_0 \oplus P_0'$

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_1' \end{pmatrix}$$

the one I construct this way?

my guess is no, but not obvious.

Corollary: take $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$

and apply $- \otimes D$, get

a long exact seq

$$\begin{aligned} \dots &\rightarrow \text{Tor}_2(L, D) \rightarrow \text{Tor}_2(M, D) \rightarrow \text{Tor}_2(N, D) \rightarrow \\ &\rightarrow \text{Tor}_1(L, D) \rightarrow \text{Tor}_1(M, D) \rightarrow \text{Tor}_1(N, D) \rightarrow \\ &\rightarrow L \otimes D \rightarrow M \otimes D \rightarrow N \otimes D \rightarrow 0 \end{aligned}$$

Corollary: TFAE:

a) A is flat

b) $\text{Tor}_{\geq 1}(A, M) = 0 \quad \forall M$

c) $\text{Tor}_1(A, M) = 0 \quad \forall M$

sim. with projective and Ext.

Pf: $a \Rightarrow b$. resolve M

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

apply $A \otimes - \rightsquigarrow$ still exact. $\text{Tor}_{\geq 1} = 0$

$b \Rightarrow c$ clear.

$c \Rightarrow a$ let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be exact

apply $A \otimes -$, get

$$\text{Tor}_1(A, N) \rightarrow A \otimes L \rightarrow A \otimes M \rightarrow A \otimes N \rightarrow 0$$

so $A \otimes -$ is an exact functor.

\square

Worksheet: Tor and torsion.