# Showing that Ext and Tor are well-defined.

Last time: given $f: M \to N$,

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

apply $- \otimes D$ still have chain map. take homology, get

$$f_* : \mathrm{Tor}_i^R(M, D) \longrightarrow \mathrm{Tor}_i^R(N, D)$$

choose different lifts $f_i' : P_i \to P_i'$

↳ they're homotopic $f_i - f_i' = ds + sd$ $\exists s$

↳ apply $- \otimes D$ still homotopic

↳ induce the same map on homology.

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**Prop**: two proj resolutions $P_\cdot \to M$ and $P_\cdot' \to M$

give the same $\mathrm{Tor}_i^R(M, -)$.

Proof:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"'] & P_1 \arrow[r] \arrow[d, "f_1"'] & P_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "1"] & 0 \\
\cdots \arrow[r] & P_2' \arrow[r] \arrow[d, "g_2"'] & P_1' \arrow[r] \arrow[d, "g_1"'] & P_0' \arrow[r] \arrow[d, "g_0"'] & M \arrow[r] \arrow[d, "1"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

apply $-\otimes D$, get maps on homology

$$\mathrm{Tor}_i^R(M,D) \xrightarrow{\bar{f}_i} \mathrm{Tor}_i^R(M,D)' \xrightarrow{\bar{g}_i} \mathrm{Tor}_i(M,D)$$

claim that $\bar{g}_i \circ \bar{f}_i = 1$

true bec. $g_i \circ f_i$ lifts $1 : M \to M$

$1 : P_i \to P_i$ also lifts $1 : M \to M$

so $g_i \circ f_i$ and $1$ induce the same map on homology. $\blacksquare$

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Soon: can resolve either $M$ or $N$

and get the same $\mathrm{Tor}_i^R(M,N)$.

**Prop** given a s.e.s. $0 \to L \to M \to N \to 0$,

$\exists$ simultaneous proj. resolutions.

**Pf**

want to construct these maps.

D&F 17.1 Prop 7 + Problem 5

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_2 \arrow[r] \arrow[d] & P_2 \oplus P_2' \arrow[r] \arrow[d] & P_2' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_1 \oplus P_1' \arrow[r] \arrow[d] & P_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r, "\binom{1}{0}"] \arrow[d, "\varepsilon"'] & P_0 \oplus P_0' \arrow[r, "(0\ 1)"] \arrow[d] & P_0' \arrow[r] \arrow[d, "\varepsilon'"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r, "\binom{1}{0}"] \arrow[d, "\varepsilon"'] \arrow[dr, "\lambda"] & P_0 \oplus P_0' \arrow[r, "(0\ 1)"] \arrow[d] & P_0' \arrow[r] \arrow[d, "\varepsilon'"] \arrow[dl, dashed, "\mu"'] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

$\lambda = f \circ \varepsilon$

$g \circ \mu = \varepsilon'$

Put $(\lambda\ \mu)$ there. why does it commute?

why is it surjective?

$$(\lambda\ \mu)\begin{pmatrix}1\\0\end{pmatrix} = \lambda = f\varepsilon \quad \checkmark$$

$$g \circ (\lambda\ \mu) = (g\circ\lambda\ \ g\circ\mu) = (\overset{0}{\cancel{gf}}\varepsilon\ \ \varepsilon') = \varepsilon' \circ (0\ \ 1)$$

Surj: by Snake lemma:

$$\begin{tikzcd}
0 \arrow[r] & \ker \varepsilon \arrow[r] & \ker \arrow[r] & \ker \varepsilon' \arrow[dddll, out=0, in=180] & \\
0 \arrow[r] & P_0 \arrow[r, "\binom{1}{0}"] \arrow[d, "\varepsilon"'] \arrow[dr, "\lambda"] & P_0 \oplus P_0' \arrow[r, "(0\ 1)"] \arrow[d] & P_0' \arrow[r] \arrow[d, "\varepsilon'"] \arrow[dl, dashed, "\mu"'] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] & N \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \operatorname{coker} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

so this is zero

and lets us continue:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "\binom{1}{0}"] \arrow[d] & P_1 \oplus P_1' \arrow[r, "(0\ 1)"] & P_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \varepsilon \arrow[r] \arrow[d] & \ker(\downarrow) \arrow[r] & \ker \varepsilon' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

repeat the same argument. $\boxed{\text{K}}$

Q: is the map $P_1 \oplus P_1' \xrightarrow{\begin{pmatrix} d_1 & 0 \\ 0 & d_1' \end{pmatrix}} P_0 \oplus P_0'$

the one I construct this way?

my guess is no, but not obvious.

Corollary: take $0 \to L \to M \to N \to 0$

and apply $- \otimes D$, get

a long exact seq

$$\cdots \to \mathrm{Tor}_2(L,D) \to \mathrm{Tor}_2(M,D) \to \mathrm{Tor}_2(N,D) \to \mathrm{Tor}_1(L,D) \to \mathrm{Tor}_1(M,D) \to \mathrm{Tor}_1(N,D) \to L\otimes D \to M\otimes D \to N\otimes D \to 0$$

Corollary: TFAE:

a) $A$ is flat

b) $\mathrm{Tor}_{\geq 1}(A,M) = 0 \quad \forall M$

c) $\mathrm{Tor}_1(A,M) = 0 \quad \forall M$

sim. with projective and Ext.

Pf: $a \Rightarrow b$. resolve $M$

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

apply $A \otimes -$ $\leadsto$ still exact. $\mathrm{Tor}_{\geq 1} = 0$

$b \Rightarrow c$ clear.

$c \Rightarrow a$ let $0 \to L \to M \to N \to 0$ be exact

apply $A \otimes -$, get

$$\cancel{\underset{0}{\mathrm{Tor}_1(A,N)}} \to A \otimes L \to A \otimes M \to A \otimes N \to 0$$

so $A \otimes -$ is an exact functor. $\boxed{}$

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Worksheet: Tor and torsion.