Last worksheet ... / $\mathbb{Z}$ for simplicity

"torsion subgroup" is left exact

$0 \to A \to B \to C \to 0$

$0 \to T(A) \to T(B) \to T(C)$ $\to$ $A \otimes \mathbb{Q}/\mathbb{Z} \to B \otimes \mathbb{Q}/\mathbb{Z} \to C \otimes \mathbb{Q}/\mathbb{Z} \to 0$

not right exact?

$0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \to \mathbb{Z}/3 \to 0$

$0 \to 0 \to 0 \to \mathbb{Z}/3$

"right der. functor" is $- \otimes \mathbb{Q}/\mathbb{Z}$

if A is torsion then

$0 \to T(A) \to T(B) \to T(C) \to 0$ is exact.

kind of a pain to show by hand?

Jon's Q about simultaneous res:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0\oplus 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "2"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "2"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "2"] \arrow[d] & \mathbb{Z}/4 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is this $2 \oplus 2$?

no: coker of $\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} \mathbb{Z}^2$

is $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \neq \mathbb{Z}/4$

Today: Ext + Tor are local.

all rings Noeth. comm. today.

Prop let $R \to S$ be a ring map, M, N be R-mods

if S is flat over R

then $\mathrm{Tor}_i^S(M\otimes S, N\otimes S) \cong \mathrm{Tor}_i^R(M,N)\otimes S$

(we'll be interested in $S = R[u^{-1}]$.

e.g. $\mathrm{Tor}_i^{R_m}(M_m, N_m) \cong \mathrm{Tor}_i^R(M,N)_m$)

Pf for RHS, take a proj. res

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \qquad (1)$$

then $\otimes_R N$, then take H_*, then $\otimes_R S$.

for LHS, take the same proj. res of M.

$\otimes S$ to get a proj. res of $M\otimes S$

$$\cdots \to P_2\otimes S \to P_1\otimes S \to P_0 \otimes S \to M\otimes S \to 0$$

still exact bec. S is flat over R.

then $\otimes_S$ with $N\otimes_R S$ (same as $\otimes_R N$)

then take H_*.

Same: take (1), $\otimes_R N$, then $\otimes_R S$, then take H_*.

does $\otimes_R S$ commute with H_*?

yes: $\otimes S$ is exact, so commutes with taking ker, im, coker so preserves ker/im $= H_*$. $\square$

Prop if M is fin gen then

$$\mathrm{Ext}^i_S(M \otimes S, N \otimes S) \cong \mathrm{Ext}^i_R(M, N) \otimes S.$$

Pf. choose a res of M by fin gen free modules (bec. R is Noetherian!)

$$\cdots \to R^{a_3} \to R^{a_2} \to R^{a_1} \to R^{a_0} \to M \to 0$$

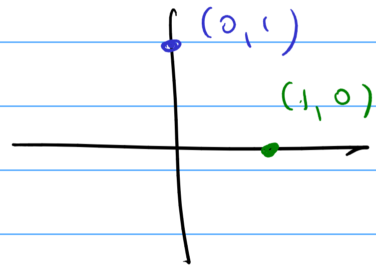
use Eisenbud Prop 2.10

$$\mathrm{Hom}_S(M \otimes S, - \otimes S) = \mathrm{Hom}_R(M, -) \otimes S$$

Same argument as above. $\square$

Examples:

$R = k[x,y]$



$I = (x-1, y)$

$J = (x, y-1)$

I and J are not projective, but

I is proj. where J is not, so

$\mathrm{Tor}^R_{\geq 1}(I, J) = 0$

Precisely: $\forall$ maximal $\underline{m} \subset R$, either

$I_{\underline{m}}$ is free or $J_{\underline{m}}$ is free (or both)

why? either x or $x-1$ is a unit in $R_{\underline{m}}$.

So $\mathrm{Tor}^R_{\geq 1}(I, J)_{\underline{m}} = \mathrm{Tor}^{R_{\underline{m}}}_{\geq 1}(I_{\underline{m}}, J_{\underline{m}}) = 0$

Similarly $\mathrm{Ext}^{\geq 1}(I, J)$ is supported at $(1, 0)$

i.e. if $\underline{m}$ is not $(x-1, y)$

then $\mathrm{Ext}^{\geq 1}(I, J)_{\underline{m}} = 0$

bec. $I_{\underline{m}} \cong R_{\underline{m}}$

In fact $\mathrm{Ext}^1(I, J) = R/I$ $\quad \mathrm{Ext}^{\geq 2} = 0$

similarly $\mathrm{Ext}^{\geq 1}(J, I)$ is supp. at $(0, 1)$

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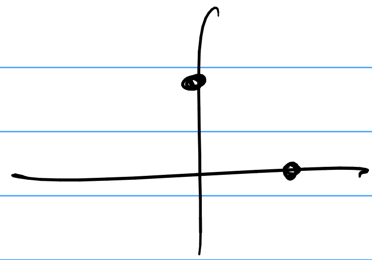
even simpler,

$$\mathrm{Tor}^R_*(R/I, R/J) = 0$$



$$\mathrm{Ext}^*_R(R/I, R/J) = 0$$

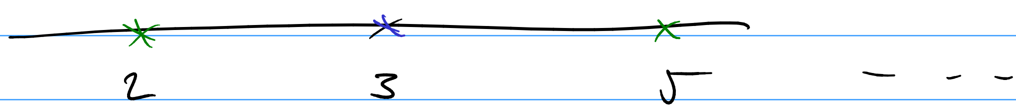
(bec. at any $\underline{m}$, either $(R/I)_{\underline{m}} = 0$

or $(R/J)_{\underline{m}} = 0$ or both.

[but we could have done this more simply with annihilators...]

$$\mathrm{Tor}^{\mathbb{Z}}_*(\mathbb{Z}/20, \mathbb{Z}/9) = 0$$

$$\mathbb{Z}/20 = \mathbb{Z}/4 \oplus \mathbb{Z}/5$$



Next goal: understand proj dim (M)

in terms of $\mathrm{Tor}_*(M, R/\underline{m})$ as $\underline{m}$ varies.

$$\text{proj. dim}(M) = \sup_{\text{max'l } \underline{m} \subset R} \sup\{ i \mid \mathrm{Tor}_i^R(M, R/\underline{m}) \neq 0 \}$$

Prop Let R be a Noeth. local ring, with max'l ideal $\mathfrak{m}$ and res. field $k = R/\mathfrak{m}$.

Let M be a fin gen module.

Then $\text{proj dim } M = \sup \{ i : \operatorname{Tor}_i^R(M, k) \neq 0 \}$.

Pf. $\geq$ is clear:
if $\exists$ proj res of length n
then $\operatorname{Tor}_{>n}(M, -)$ vanishes.

if RHS $= \infty$ then nothing to prove.

if RHS $< \infty$, prove LHS $\leq$ RHS by induction.

Base case: if $\operatorname{Tor}_{\geq 1}^R(M, k) = 0$ then M is proj.

did it before: take minimal generators

$$0 \to \ker \to R^n \to M \to 0$$

apply $- \otimes k$:

$$0 \to \cancel{\operatorname{Tor}_1(M, k)} \to \ker \otimes k \to k^n \xrightarrow{\sim} M \otimes k \to 0$$

so $\ker \otimes k = 0$, so $\ker = 0$ by Nakayama.

Induction: again take

$$0 \to \ker \to R^n \to M \to 0$$

$$\begin{aligned}\text{proj dim } M &\le \text{proj dim}(\ker) + 1 \\ &\le \max\{i \mid \mathrm{Tor}_i(\ker, k) \neq 0\} + 1 \\ &= \max\{i \mid \mathrm{Tor}_i(M, k) \neq 0\}\end{aligned}$$

inductive hyp.

why the last $=$?

$$\cdots \to \mathrm{Tor}_2(\ker, k) \to 0 \to \mathrm{Tor}_2(M, k) \to \mathrm{Tor}_1(\ker, k) \to 0 \to \mathrm{Tor}_1(M, k) \to \ker \otimes k \xrightarrow{0} k^n \xrightarrow{\cong} M \otimes k \to 0$$

$$\mathrm{Tor}_i(M, k) = \mathrm{Tor}_{i-1}(\ker, k) \quad \forall \; i \ge 1.$$

$\blacksquare$