R still Noeth. comm.
M still fin gen

last time: let $\underline{m} \subset R$ be a max'l ideal
then $\text{proj. dim.}_{R_{\underline{m}}}(M_{\underline{m}})$

$$= \sup\{ i \mid \text{Tor}_i^{R_{\underline{m}}}(M_{\underline{m}}, \underbrace{R_{\underline{m}}/\underline{m}R_{\underline{m}}}_{\text{residue field}}) \neq 0 \}$$

$$= \sup\{ i \mid \text{Tor}_i^R(M, R/\underline{m}) \neq 0 \}$$

Prop $\text{proj. dim}_R(M) = \sup_{\underline{m} \subset R} \text{proj dim}_{R_{\underline{m}}}(M_{\underline{m}})$.

$$= \sup_{\underline{m} \subset R} \text{ of this}$$

Pf $\geq$ is easy:
take a proj. res

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

localize at $\underline{m}$ $\rightsquigarrow$ get a proj. res of $M_{\underline{m}}$

$\leq$ is clear if RHS $= \infty$.

otherwise let $N =$ RHS

let $\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$
be any proj. res of M.

break into short exact seqs.

$0 \to K_0 \to P_0 \to M \to 0$ where $K_0 = \ker \varepsilon$

$0 \to K_1 \to P_1 \to K_0 \to 0$ where $K_1 = \ker d_1$

$0 \to K_2 \to P_2 \to K_1 \to 0$ etc

apply $- \otimes R/m$

$\cdots \to \mathrm{Tor}_2(K_0, R/m) \to 0 \to \mathrm{Tor}_2(M, R/m)$

$\to \mathrm{Tor}_1(K_0, R/m) \to 0 \to \mathrm{Tor}_1(M, R/m)$

$\to K_0 \otimes R/m \to P_0 \otimes R/m \to M \otimes R/m \to 0$

see that $\mathrm{Tor}_i(K_0, R/m) = \mathrm{Tor}_{i+1}(M, R/m) \ \forall i \geq 1$

Similarly $\mathrm{Tor}_i(K_1, R/m) = \mathrm{Tor}_{i+1}(K_0, R/m)$

$= \mathrm{Tor}_{i+2}(K_0, R/m) \ \forall i \geq 1$

by hypothesis, $\mathrm{Tor}_{\geq N}(M, R/m) = 0$

thus $\mathrm{Tor}_{\geq 1}(K_{N-1}, R/m) = 0$

thus K_{n-1} is proj.

why? bec. if localize at any $\underline{m}$

then it's proj. by Friday's results.

$\mathrm{Tor}_{\geq 1}((K_{n-1})_{\underline{m}}, \text{res. field}) = 0 \quad \forall \underline{m}$

so $(K_{n-1})_{\underline{m}}$ is proj. / $R_{\underline{m}}$ $\forall \underline{m}$

now here's a proj. res of length n!

$$0 \to \underbrace{K_{n-1} \to P_{n-1} \to P_{n-2} \to \cdots \to P_0}_{\text{length } n, \text{ counting from } 0} \to M \to 0$$

$\boxtimes$

This was ess. the Hilbert Syzygy thm.

Recall: glob dim $R = \sup_M$ proj dim M

Thm (Auslander '55): can take sup over just fin gen modules

Read pf in Eisenbud A3.18 p. 650

we have seen:

$$\operatorname{glob\,dim} R = \sup_{\underline{m}} \operatorname{glob\,dim} R_{\underline{m}}$$

$$\operatorname{glob\,dim} R_{\underline{m}} = \operatorname{proj\,dim} R/\underline{m}$$

examples:

- $\operatorname{glob\,dim} \mathbb{Z} = 1$.

  $\operatorname{proj\,dim}_{\mathbb{Z}} \mathbb{Z}/p = 1$: $\quad 0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$

- $\operatorname{glob\,dim} k[x] = 1$ (if $k = \mathbb{C}$ certainly)

  $$0 \to R \xrightarrow{x-a} R \to R/_{x-a} \to 0$$

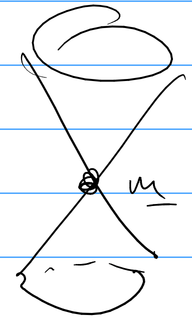
- $\operatorname{glob\,dim} k[x,y] = 2$

  Koszul res.

  $$0 \to R \to R^2 \to R \to R/_{(x-a,\, y-b)} \to 0$$

- Soon: $\operatorname{glob\,dim} k[x_1 \dots x_n] = n$.

- $\operatorname{glob\,dim} k[x,y,z]/_{x^2+y^2=z^2} = \infty$



because if $\underline{m} = (x,y,z)$

then $\operatorname{Tor}_i(R/\underline{m}, R/\underline{m}) \neq 0 \quad \forall i$.