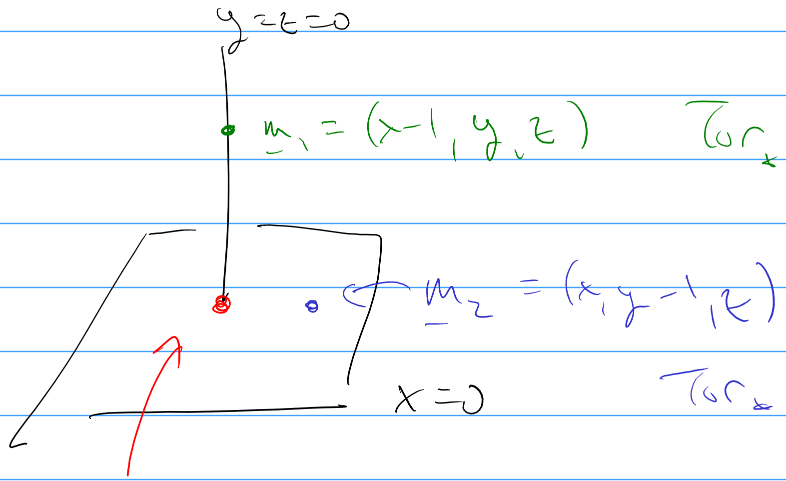
on last worksheet,

$$R = k[x,y,z]\big/(xy,\, xz)$$



$\mathrm{Tor}_*\left(R/\underline{m}_1,\, R/\underline{m}_1\right) = 1, 1, 0 \ldots$

$\mathrm{Tor}_* = 1, 2, 1, 0, \ldots$

$\underline{m}_3 = (x, y, z)$

$\mathrm{Tor}_* = 1, 3, 5, 8, 13, 21, \ldots$ (Fibonacci!)

$R_{\underline{m}_1}$ is the same as $k[x]_{(x-1)}$

because in $R_{\underline{m}_1}$, x is a unit

$$k[x,y,z]\big/(xy,\, xz) \rightsquigarrow k[x,y,z]\big/(y,z) \cong k[x].$$

could have computed

$$\mathrm{Tor}_*^{k[x]}\left(\frac{k[x]}{x-1}, \frac{k[x]}{x-1}\right) = 1, 1, 0, 0 \ldots$$

easily using $0 \to k[x] \xrightarrow{x-1} k[x] \to k[x]\big/_{x-1} \to 0$

$R_{\underline{m}_2} = k[y,z]_{(y-1,z)}$

because y is a unit in $R_{\underline{m}_2}$

$k[x,y,z]/(xy, xz) \rightsquigarrow k[x,y,z]/(x) \cong k[y,z]$

Could have computed

$\mathrm{Tor}_*^{k[y,z]}\left(\frac{k[y,z]}{(y-1,z)}, \frac{k[y,z]}{(y-1,z)} \right) = 1, 2, 1, 0, 0, \ldots$

easily with a 2-step Koszul complex
that we've studied.

$R_{\underline{m}_3}$ is really as complicated as
it looks in Macaulay 2.

Corey points out: that isn't a field
every PID has glob dim. 1
why? let $m \subset R$ be maximal.
write $\underline{m} = (r)$ $r \neq 0$

then $0 \to R \xrightarrow{r} R \to R/\underline{m} \to 0$

so proj. $\dim_R (R/\underline{m}) = 1$.

so $k[x]$, $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$

all have glob dim 1.

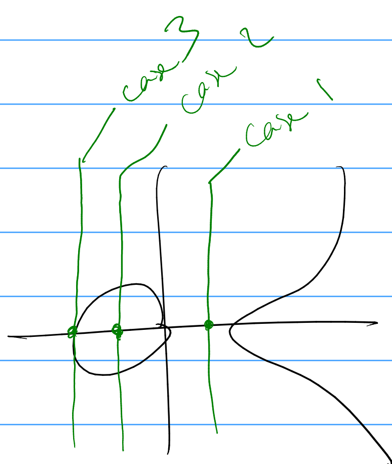
but $\mathbb{Z}[\sqrt{-5}]$ is not a PID
and still has glob. dim 1.

<u>Details</u>

$$\mathbb{Z} \longrightarrow R = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[y]/_{y^2=-5}$$

geometric analogy:

$$\mathbb{R}[x] \longrightarrow \mathbb{R}[x,y]/_{y^2=x^3-x}$$



Claim is that $\mathrm{Tor}^R_{\geq 2}\left(R/\mathfrak{m},\ R/\mathfrak{m}\right) = 0$

$\forall$ max'l $\mathfrak{m} \subset R$.

what do $\mathfrak{m}$s look like?

consider $\mathfrak{m} \cap \mathbb{Z} = p\mathbb{Z}$ for some prime p.

$pR \subset \mathfrak{m}$

Compute $\mathrm{Tor}_i^R(R/p, R/p) = \begin{cases} R/p & i=0 \\ R/p & i=1 \\ 0 & i \geq 2 \end{cases}$

(use $0 \to R \xrightarrow{p} R \to R/p \to 0$)

saw that $R = \mathbb{Z}[y]/y^2+5$

so $R/p = \mathbb{F}_p[y]/y^2+5$

Case 1: if y^2+5 is irred
then R/p is a field ($\mathbb{F}_{p^2}$)
so pR is max'l
so $pR = \underline{m}$.

Case 2: if $y^2+5 = (y-a)(y-b)$ in $\mathbb{F}_p[y]$

then $R/p = \mathbb{F}_p[y]/(y-a)(y-b)$

$= \mathbb{F}_p[y]/y-a \oplus \mathbb{F}_p[y]/y-b = \mathbb{F}_p \oplus \mathbb{F}_p$

max'l ideals of R over pR
$\longleftrightarrow$ max'l ideals of R/pR.

two of them $\longleftrightarrow$ $y-a$ $\quad$ $y-b$

one is $\underline{m}$, call the other $\underline{m}'$.

$$R/\underline{p} = R/\underline{m} \oplus R/\underline{m}'$$

$$\mathrm{Tor}_*(R/\underline{p}, R/\underline{p}) = \mathrm{Tor}(R/\underline{m}, R/\underline{m}) \oplus \underbrace{\cancel{\mathrm{Tor}(R/\underline{m}, R/\underline{m}')}}_{0}$$
$$\oplus \underbrace{\cancel{\mathrm{Tor}(R/\underline{m}', R/\underline{m})}}_{0} \oplus \mathrm{Tor}(R/\underline{m}', R/\underline{m}')$$

this vanishes above deg 1, so these do too.

Case 3: $y^2+5 = (y-a)^2$ in $\mathbb{F}_p[y]$

think... either $p=2$ or $p=5$

if $p=5$, $R/p = \mathbb{F}_5[y]/_{y^2+5} = \mathbb{F}_5[y]/_{y^2}$

max'l ideals over pR
$\longleftrightarrow$ max'l ideals over 0 in there
just (y)

$\underline{m} = (\sqrt{-5})$ principal $\Rightarrow$ $R/\underline{m}$ proj. dim 1.

$$0 \to R \xrightarrow{\sqrt{-5}} R \to R/\underline{m} \to 0$$

if $p=2$ then $R/p \cong \mathbb{F}_2[y]/_{y^2+5} = \mathbb{F}_2[y]/_{(y+1)^2}$

max'l ideals over $2R \longleftrightarrow (y+1)$

so $\underline{m} = (2, 1+\sqrt{-5})$

not a principal ideal
but it's proj. as an R-module

$$0 \to \underline{m} \to R \to R/\underline{m} \to 0$$

so proj. dim $R/\underline{m} = 1$.

Worksheet: $\mathbb{Z}[\sqrt{-3}]$.

geom analogue: $\mathbb{R}[x] \longrightarrow \mathbb{R}[x,y]/_{y^2 = x^3+x^2}$