Last time: $\mathbb{Z}[\sqrt{-5}]$
not a PID but
still glob dim $= 1$.

could have said:
$\forall\, \underline{m} \subset R$, $R_{\underline{m}}$ is a PID.

Worksheet: $R = \mathbb{Z}[\sqrt{-3}]$

$\underline{m} = (2, 1+\sqrt{-3})$

found that

$\mathrm{Tor}_*(R/\underline{m}, R/\underline{m}) = 1\ 2\ 2\ 2\ 2 \ldots$

glob dim $= \infty$.

Today: Koszul complexes and

$\text{gl dim } k[x_1, \dots x_n] = n.$

You saw: for $R = k[x,y]$

$\underline{m} = (x,y)$

this is a res of $R/\underline{m}$:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{(x \ y)} R \longrightarrow R/\underline{m} \to 0$$

so $\mathrm{Tor}^R_*(R/\underline{m}, R/\underline{m}) = 1, 2, 1, 0 \dots$

What about $n > 2$?

let $R = k[x_1, \dots, x_n]$

$M = R^n$ with basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$

(just symbols; or M = vector fields on k^n)

$\xi = x_1 \frac{\partial}{\partial x_1} + \dots + x_n \frac{\partial}{\partial x_n}$ "Euler vector field"

$M^* = R^n$ w/ dual basis $dx_1 \dots dx_n$

(just symbols; or 1-forms)

pairing betw. M and M^*:

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Koszul complex:

$$0 \to \Lambda^n M^* \to \dots \to \Lambda^k M^* \to \Lambda^{k-1} M^* \to \dots \to \Lambda^2 M^* \to M^* \to \underline{R} \to 0$$

$$\omega \longmapsto \xi \lrcorner \omega \text{ or } i_\xi \omega$$

(for contracting / interior mult with a v.f.: read Jack Lee, Smooth Manifolds, Ch. 14 of 2nd ed.)

$\omega \longmapsto \xi \lrcorner \omega$ is an R-lin. map

determined by $dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Lambda^k M^*$

$$\downarrow$$

$$\sum_{j=1}^{k} (-1)^{j+1} x_{i_j} \, dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k}$$

Claim: Koszul complex is a res of $R/(x_1, \dots, x_n)$

not surprising: the map $M^* = R^n \longrightarrow R$ is just $(x_1 \cdots x_n)$

so coker $= R/(x_1, \dots, x_n)$

but why is it exact elsewhere?

define $d: \Lambda^k M^* \to \Lambda^{k+1} M^*$

k-linear but not R-linear.

$$d(f \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

exactness follows from

$$d(\xi \lrcorner \omega) + \xi \lrcorner d\omega = \omega.$$

corrected next time.

do it by hand for $n=3$, $k=2$

$$\omega = f\, dx \wedge dy$$

$$d\omega = \frac{\partial f}{\partial x} \cancel{dx \wedge dx \wedge dy} + \frac{\partial f}{\partial y} \cancel{dy \wedge dx \wedge dy} + \frac{\partial f}{\partial z} dz \wedge dx \wedge dy$$

$$= f_z\, dx \wedge dy \wedge dz$$

$$\xi \lrcorner \text{that} = \cancel{x f_z\, dy \wedge dz} - \cancel{y f_z\, dx \wedge dz} + z f_z\, dx \wedge dy$$

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$$\xi \lrcorner \omega = xf\, dy - yf\, dx$$

$$d(\text{that}) = (f + x f_x)\, dx \wedge dy + \cancel{x f_y\, dy \wedge dy}\ \ \cancel{x f_z\, dz \wedge dy})$$

$$- \cancel{y f_x\, dx \wedge dx} - (f + y f_y)\, dy \wedge dx - \cancel{y f_z\, dz \wedge dx}$$

$$\xi \lrcorner d\omega + d(\xi \lrcorner \omega) = (2f + x f_x + y f_y + z f_z)\, dx \wedge dy$$

if f is homog of deg e

then $x f_x + y f_y + z f_z = e \cdot f$

$$= (2+e) \cdot \omega \; ?\,?$$

fix it next time, sorry.

worksheet: understand differentials by hand when $n=3$.