Last time: $R = k[x_1, \dots, x_n]$

$M = R^n$ with basis $\partial/\partial x_1, \dots, \partial/\partial x_n$

Euler v.f. $\xi \in M$

$M^* = R^n$ with basis $dx_1, \dots, dx_n$

Koszul complex

$$0 \to \Lambda^n M^* \longrightarrow \cdots \to \Lambda^2 M^* \xrightarrow{\xi \lrcorner} M^* \xrightarrow{\xi \lrcorner} R \longrightarrow 0$$

homology here is $R/(x_1, \dots, x_n)$

but exact elsewhere.

if $\omega = f\, dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \Lambda^k M^*$

where f is homog. of deg e,

then $d(\xi \lrcorner \omega) + \xi \lrcorner d\omega = (k+e)\omega$

[Cartan's magic formula: it's $\mathcal{L}_\xi \omega$.

$\mathcal{L}_\xi f = \xi f = ef$

$\mathcal{L}_\xi dx_i = dx_i$

$\mathcal{L}_\xi$ obeys a Leibniz rule.]

Let $\omega \in \Lambda^k M^*$ be general. $k \geq 1$

write ω as a sum of homog. pieces:

$\omega = \omega_0 + \omega_1 + \cdots + \omega_e + \cdots + \omega_\ell$

if $\xi \lrcorner \omega = 0$ then each $\xi \lrcorner \omega_e = 0$

so $\xi \lrcorner d\omega_e = (k+e)\, \omega_e$

so $\xi \lrcorner \left(\frac{1}{k} d\omega_0 + \frac{1}{k+1} d\omega_1 + \cdots + \frac{1}{k+\ell} d\omega_\ell \right) = \omega.$

So the complex is exact where claimed.

(this proof requires char $k = 0$,
but the conclusion is true more generally.)

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if $R = k[x_1 \ldots x_n]$ and $\underline{m} = (x_1, \ldots, x_n)$

then $\mathrm{Tor}^R_*(R/\underline{m}, R/\underline{m}) = 1, n, \binom{n}{2}, \ldots, n, 1, 0$ —

resolve by koszul cx.
$\otimes R/m$ w/ all differentials $= 0$

proj. dim $R/\underline{m} = n$.

If $m = (x_1 - a_1, \dots, x_n - a_n)$

then it's similar: proj dim $R/m = n$.

If $k = \bar{k}$, that proves that

glob dim $R = n$.

[An answer to question:
$R = k[x, y, z, w]$ $\quad I = (xz - y^2, xw - yz, yw - z^2)$

Koszul cx of then 3 is not exact.]

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if $k \neq \bar{k}$, still glob dim $k[x_1 \dots x_n] = n$

know glob dim $\geq n$

because of $\mathrm{Tor}_*(R/(x_1 \dots x_n), \text{self})$.

Let's prove $\leq$.

$R = k[x_1 \dots x_n]$ $\quad S = \bar{k}[x_1, \dots, x_n] = R \otimes_k \bar{k}$

Claim: S is flat over R

Better: a seq of R-modules

$0 \to L \to M \to N \to 0$ is exact

iff

$0 \to L \otimes_R S \to M \otimes_R S \to N \otimes_R S \to 0$ is exact

"faithfully flat"

Pf $0 \to L \to M \to N \to 0$

is exact as R-mod

iff it's exact as k-vector spaces.

iff $0 \to L \otimes_k \bar{k} \to M \otimes_k \bar{k} \to N \otimes_k \bar{k} \to 0$

is exact as $\bar{k}$-vector spaces.

(think about it)

Same

iff $0 \to L \otimes_R S \to M \otimes_R S \to N \otimes_R S \to 0$

is exact as S-modules. $\boxtimes$

Prop if $R \to S$ is faithfully flat

then an R-module A is flat$/R$

$\iff$ $A \otimes_R S$ is flat$/S$.

Pf $\Rightarrow$ easy. for $\Leftarrow$,

let $0 \to L \to M \to N \to 0$ be

an exact seq. of R-mods.

is $0 \to L \otimes_R A \to M \otimes_R A \to N \otimes_R A \to 0$ exact?

yes iff same $\otimes_R S$ is exact, but

$- \otimes_R A \otimes_R S = - \otimes_R S \otimes_S (A \otimes_R S)$ and both those

operations are exact by hypothesis. $\boxtimes$

Prop let M be a fin gen module over $R = k[x_1, \ldots x_n]$.

then proj dim $M \leq n$.

$S = \bar{k}[\text{——}]$

Pf take any fin. gen. proj. res.

$$\cdots \to P_1 \to P_0 \to M \to 0$$

cut it off: let $K = \ker d_{n-1}$:

$$0 \to K \to P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \to P_0 \to M \to 0$$

claim: K is proj.

know $K \otimes_R S$ is proj, because

$$\mathrm{Tor}^S_1(K \otimes_R S, S/\underline{m}) = \mathrm{Tor}^S_{>n}(M \otimes_R, S/\underline{m}) = 0$$

so $K \otimes_R S$ is flat

so K is flat

so K is proj. (bec. fin gen). ∎