Goals:

- Most of the way through proving this thm of Serre:
  if R is Noeth. comm.
  then glob dim $R < \infty$ iff R is <u>regular</u>
  in which case glob dim = Krull dim.

- Ext and extensions

- Tor and Ext can be computed by resolving either factor
  feat. spectral sequences

- Thing from Day 1:
  if R is Noeth. comm.
  and $I \subset R$ locally gen by a regular seq.
  and $J \subset R$ such that R/J is Cohen-Macaulay,
  you hope that $\operatorname{codim} I + \operatorname{codim} J = \operatorname{codim}(I+J)$

  thm: # of non-vanishing $\operatorname{Tor}_i^R(R/I, R/J)$

  tells the difference.

- need to talk about <u>depth</u>
  and its rel'n to Ext.

Let R be a Noeth. local ring.
$\underline{m} \subset R$ max'l ideal
$k = R/\underline{m}$ residue field

Know: glob dim R = proj dim k
= # non-zero $\mathrm{Tor}_i^R(k,k)$

let $r_1, \dots, r_n \in \underline{m}$ be minimal generators.
(seen: $n = \dim_k (\underline{m} \otimes_R k)$
which $= \dim_k \mathrm{Tor}_1^R(k,k)$.
why? take $0 \to \underline{m} \to R \to k \to 0$
apply $- \otimes k$ to get
$0 \to \mathrm{Tor}_1(k,k)$
$\to \underline{m} \otimes k \to k \to k \to 0$)

from Krull's principal ideal thm,
know that dim $R \leq n$.
(maybe say something later.)

foreshadowing: if Krull dim $= n$ then
glob dim $= n$ and R is regular

if Krull dim $< n$ then
glob dim $= \infty$ and R is not regular.

we can make a Koszul complex out of $r_1, \dots, r_n$:

$$0 \to R \to R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \xrightarrow{(r_1 \dots r_n)} R \to R/\mathfrak{m} \to 0$$

If it's exact, then it gives

$\dim_k \operatorname{Tor}_i^R(k,k) = \binom{n}{i}$

so glob dim $R = n$.

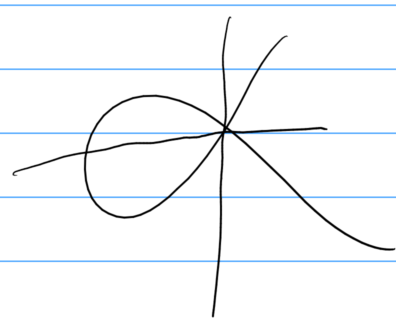
Examples:

- in $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$, it's exact.
- in $\mathbb{Z}_{(p)}$, it's exact

(more gen. in any local PID).

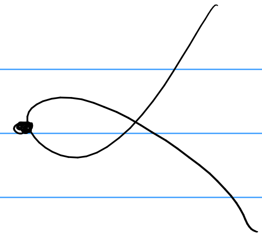
$\mathfrak{m} = (p)$ Koszul complex $0 \to R \xrightarrow{p} R \to \mathbb{Z}/p \to 0$

- in $k[x,y]/_{y^2 = x^3 + x^2}$



$y^2 = x^2(x+1)$

if we localize at $(x+1, y)$



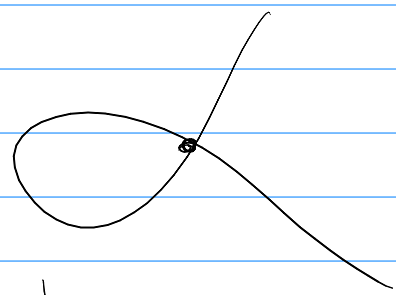
then the maximal ideal is still called $(x+1, y)$, but now $x+1 = \frac{y^2}{x^2}$,

so with min'l generators it's (y)

so Kosz. complex $0 \to R_{\mathfrak{m}} \xrightarrow{y} R_{\mathfrak{m}} \to k \to 0$

is exact.

if we localize at (x, y)



then max'l ideal (x,y) really needs 2 generators, but

$$0 \to R_{\mathfrak{m}} \xrightarrow{(y \ \ -x)} R_{\mathfrak{m}}^2 \xrightarrow{\binom{x}{y}} R_{\mathfrak{m}} \to k \to 0$$

is not exact. $\hookrightarrow H_1 = k$

- in $\mathbb{Z}[\sqrt{-5}]$ loc. at $(2, 1+\sqrt{-5})$

we have $2 = \dfrac{(1+\sqrt{-5})(1-\sqrt{-5})}{3}$

so max'l ideal is $(1+\sqrt{-5})$

$0 \to R_{\mathfrak{m}} \xrightarrow{1+\sqrt{-5}} R_{\mathfrak{m}} \to \mathbb{Z}/2 \to 0$ is exact

. in $\mathbb{Z}[\sqrt{-3}]$ loc at $(2, 1+\sqrt{-3})$

the max'l ideal really needs 2 gens.

(you found that $\mathrm{Tor}_1(k,k) = k^2$)

but

$$0 \to R_{\mathfrak{m}} \xrightarrow{\begin{pmatrix} 1+\sqrt{-3} \\ -2 \end{pmatrix}} R_{\mathfrak{m}}^2 \xrightarrow{(2 \;\; 1+\sqrt{-3})} R_{\mathfrak{m}} \to \mathbb{Z}/2 \to 0$$

is not exact. $\hookrightarrow H_1 = \mathbb{Z}/2$.

Going for: Koszul complex will be exact iff those gens of $\mathfrak{m}$ form a regular seq.

Def. Let R be a Noeth. comm. ring.

a seq. $r_1, \ldots, r_n \in R$ is a <u>regular seq</u>

if $(r_1, \ldots, r_n)$ is not the unit ideal

and each r_i is not a zero-div. in $R/(r_1, \ldots, r_{i-1})$

Def a Noeth. local ring is regular

if the max'l ideal is gen.
by a regular seq.

a Noeth. comm ring is regular if
every $R_{\mathfrak{p}}$ is regular.
(maybe every $R_{\mathfrak{m}}$ is enough?)

Thm (next time):

if $r_1 \dots r_n$ is a reg. seq,

then $H_{>0}(\text{Koszul cx of } r_1 \dots r_n) = 0$

if R is local and $r_1 \dots r_n \in \mathfrak{m}$

then converse holds too.