R a Noeth. comm. ring.
$r_1, \dots, r_n \in R$ is a <u>regular sequence</u>

if $(r_1, \dots, r_n)$ is not the unit ideal

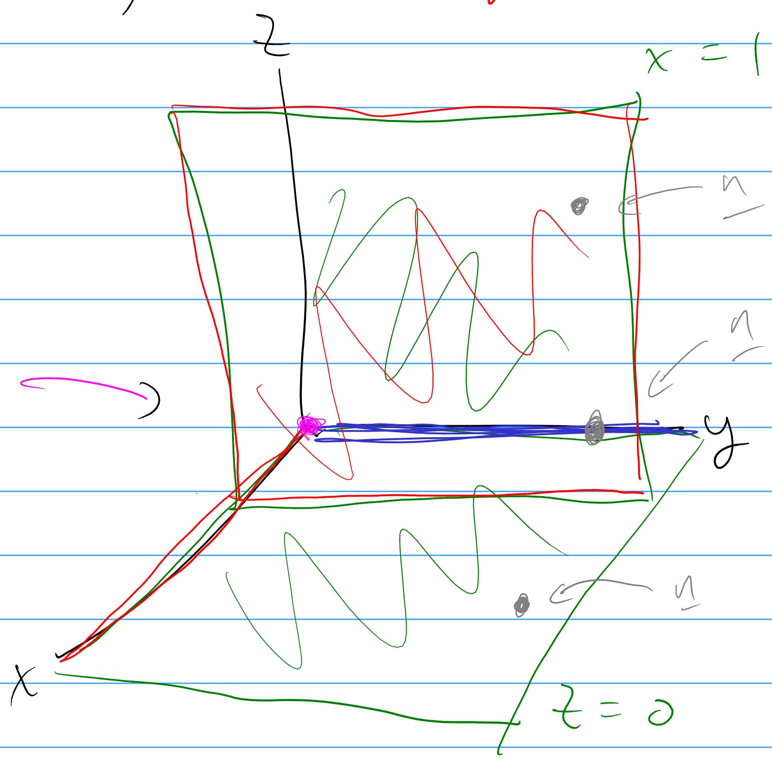
and each r_i is not a zero-div.
in $R/(r_1, \dots, r_{i-1})$

Last worksheet:

$$R = k[x,y,z]/(x-1)z$$

seq: x, $(x-1)y$

$\underline{m} = (x, (x-1)y)$
$= (x, y)$
$= (x, y, z)$



In $R_{\underline{m}} = k[x,y,z]_{(x,y,z)} \Big/ \cancel{(x-1)}z$

unit if we've localized at (x,y,z)

$= k[x,y]_{(x,y)}$

now our sequences are x, unit$\cdot y$

or unit$\cdot y$, x

which is regular in either order.

Localize at another max'l ideal $\underline{n}$,

then either $x \notin \underline{n}$ or $(x-1)y \notin \underline{n}$

(else $\underline{m} \subset \underline{n}$ so $\underline{m} = n$)

so one is a unit

and it's not a reg. seq. for that reason.

In any case, Koszul complex has $H_{\geq 1} = 0$

<u>Thm</u> let R be a Noeth. local ring

let $r_1, \dots, r_n \in \underline{m}$ [equiv: ideal$(r_1 \dots r_n) \neq (1)$]

K = their koszul complex

$$\overset{n^{th}\text{ place}}{\overset{\downarrow}{R}} \to R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \xrightarrow[(r_1 \cdots r_n)]{} \overset{0^{th}\text{ place}}{\overset{\downarrow}{R}}$$

observe: $H_0(K) = R/(r_1, \dots, r_n) \neq 0$

(a) if $H_i(K) = 0$ then $H_{>i}(K) = 0$

(b) $r_1, \dots, r_n$ is a regular seq.

$\Downarrow$ if R isn't local, get this direction only

iff $H_1(K) = 0$

<u>Pf</u> (a) is by induction on n.

base case $n=1$: if $H_1 = 0$ then there is no $H_{>1}$ to check.

inductive step: let

L = koszul cx of $r_1, \dots, r_{n-1}$.

get an exact seq of kosz. complexes

$$L \to K \to L[1]$$

$$\left(\begin{array}{cccccccccc}
 & & L & & K & & L[1] & & & \\
\text{e.g.} & 0 \to & 0 & \to & R & \to & R & \to 0 & & \text{if } n=3 \\
 & & \downarrow & & \downarrow & & \downarrow & & & \\
 & 0 \to & R & \to & R^3 & \to & R^2 & \to 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & \\
 & 0 \to & R^2 & \to & R^3 & \to & R & \to 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & \\
 & 0 \to & R & \to & R & \to & 0 & \to 0 & &
\end{array}\right)$$

so exact seq. on homology

$$\begin{array}{l}
H_3(L) \to H_3(K) \to H_2(L) \xrightarrow{\cdot r_n} \\
\to H_2(L) \to H_2(K) \to H_1(L) \xrightarrow{\cdot r_n} \\
\to H_1(L) \to H_1(K) \to H_0(L) \xrightarrow{\cdot r_n} \\
\to H_0(L) \to H_0(K) \to 0
\end{array}$$

boundary maps are $\cdot r_n$ ... read about it.

now e.g. if $H_2(K) = 0$

then $H_2(L) / r_n H_2(L) = 0$

So $H_2(L) / \underline{m} H_2(L) = 0$

So $H_2(L) = 0$ by Nakayama's lemma.

so $H_{\geq 2}(L) = 0$ by inductive hyp.

so $H_{\geq 2}(K) = 0$ by long exact seq.

(6) also by induction.

for $n=1$, the Kosz complex is

$$\underset{1}{R} \xrightarrow{r_1} \underset{0}{R}$$

$H_1 = 0$ iff $\cdot r_1$ is inj.

iff r_1 is not a zero-div.

inductive step:
take same $L = \mathrm{Kosz}(r_1 \cdots r_{n-1})$

Consider

$$H_1(L) \xrightarrow{r_n} H_1(L) \to H_1(K) \to H_0(L) \xrightarrow{r_n} H_0(L) \to H_0(K) \to 0$$

if $r_1 \cdots r_n$ is a reg. seq then

$r_1 \cdots r_{n-1}$ is reg, so $H_1(L) = 0$

and $R/_{r_1, \cdots, r_{n-1}} \xrightarrow{r_n} R/_{r_1, \cdots, r_{n-1}}$

is injective.
thus $H_1(K) = 0$

Conversely, if $H_1(K)=0$
then $H_1(C)/_{r_n H_1(C)} = 0$

so $H_1(C)=0$ as before,

so $r_1 \dots r_{n-1}$ is reg, and

$H_0(C) \xrightarrow{r_n} H_0(C)$ is inj,

so r_n is not a zero-div on $R/_{r_1 \dots r_{n-1}}$.

$\boxtimes$

Corollary: in a local ring,

a regular seq is regular in any order.

Pf: permutation induces an iso
of the Koszul complexes.

Next time: my favorite fact.

let R be a Noeth. local ring,

$r_1 \dots r_n \in \underline{m}$ $\qquad K =$ Kosz. complex

let k be the greatest int. such that
$H_k(K) \neq 0$.

then any maximal regular sequence
in the ideal $(r_1, \dots, r_n) =: I$
has length $n-k$.

In particular they all have the same length.

Foreshadowing: that length $\overset{n-k}{\vee}$ is called the <u>depth</u> of I.

in a regular local ring, $\underbrace{\text{depth} = \text{codimension}}_{\text{"Cohen-Macaulay"}}$.

Also: depth = smallest l such that
$\operatorname{Ext}^l(R/I, R) \neq 0$.