Working toward:
understanding how Tor, Ext, proj. dim
interact with geometric ideas of dim.

Warning: my indexing doesn't match Eisenbud's

R = Noeth. local ring

$r_1, \dots, r_n \in \underline{m}$

K = Koszul complex

$$0 \to R \to R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \xrightarrow{(r_1 \cdots r_n)} R \to 0$$

this course: H_n — — — H_2 H_1 H_0

Eisenbud: H^0 H^1 — — — — — H^n

in my research life: H^{-n} — — — H^{-2} H^{-1} H^0

last time: $r_1, \dots, r_n$ is a reg. seq.

iff $H_{\geq 1}(K) = 0$

iff $H_1(K) = 0$

Thm: let k be the greatest int. s.t. $H_k(K) \neq 0$ but $H_{>k} = 0$

then any maximal reg seq. in

$I := (r_1, \dots r_n)$

has length $\underbrace{n-k}$.

↪ Def: this is called the **depth** of I

wants to be codim I

Lemma: every r_i acts on every $H_j(K)$ by zero. hence every element of I does.

believable: $H_0(K) = R/I \dots$

claim is that the rest are R/I-modules.

Pf of Lemma:

$r_1, \dots, r_n$ give a map

$$R[x_1 \dots x_n] \xrightarrow{\varphi} R$$
$$x_i \longmapsto r_i$$

Kosz. complex of $x_1 \dots x_n \in R[\vec{x}]$ is exact.

apply $\otimes_\varphi R$, get K.

thus $H_i(K)$ computes $\mathrm{Tor}_i^{R[\vec{x}]}\left(R[\vec{x}]/\vec{x}, R\right)$

could also have resolved R as an $R[\vec{x}]$-mod and applied $\otimes R[\vec{x}]/\vec{x}$

which would kill all $\varphi(x_i) = r_i$ $\square$

Pf of thm Know $H_k(K) \neq 0$, $H_{>k} = 0$

let $y_1, \dots, y_s \in I$ be a maximal reg seq.

consider $\mathrm{Kosz}(r_1, \dots, r_n, y_1) =: K_1$

$K \to K_1 \to K[1]$ exact seq. of complexes

$$\to H_i(K) \to H_i(K_1) \to H_{i-1}(K) \xrightarrow{y_1} H_{i-1}(K) \to \cdots$$ y_1 acts by 0

so $H_{k+1}(K_1) \neq 0$ and $H_{>k+1} = 0$

for $\mathrm{Kosz}(r_1 \dots r_n, y_1, y_2)$

$H_{k+2} \neq 0$ and $H_{>k+2} = 0$

for $\mathrm{Kosz}(r_1 \dots r_n, y_1 \dots y_s)$

$H_{k+s} \neq 0$ and $H_{>k+s} = 0$

other way:

$\mathrm{Kosz}(y_1 \dots y_s)$ has $H_0 \neq 0$ $H_{>0} = 0$

because $y_1 \dots y_s$ is a reg. seq.

want to say:

$\mathrm{Kosz}(y_1 \dots y_s, r_1)$ has $H_1 \neq 0$ $H_{>1} = 0$

if $H_1 = 0$ then $y_1, \dots, y_s, r_1$ is regular

so $y_1 \dots y_s$ was not a maximal reg. seq.

what is H_1 ?

$$\cancel{H_1(K(\underline{y}))}^{0} \to H_1((K(\underline{y}, r_1)) \to H_0(K(\underline{y})) \xrightarrow{\cdot r_1} H_0(K(\underline{y})) \to H_0(K(\underline{y}, r_1)) \to 0$$

let $S = R/_{y_1 \dots y_s}$. then

$$H_1((K(\underline{y}, r_1)) = \{ z \in S \mid r_1 z = 0 \}$$

plausible:

in $\mathrm{Kosz}(y_1, \dots, y_s, r_1 \dots r_n)$,

$$H_n = \{ z \in S \mid r_1 z = r_2 z = \dots = r_n z = 0 \} \qquad H_{>n} = 0$$

assume for now that if $y_1, \dots, y_s$ was maximal then that $H_n \neq 0$

so $H_n(\mathrm{Kosz}(\vec{y}, \vec{r})) \neq 0$ $\quad H_{>n} = 0$

saw above that $H_{k+s} \neq 0$ and $H_{>k+s} = 0$

conclude that $k+s = n$ so $s = n-k$.

we've reduced to this claim:

$\bar{r}_1, \dots, \bar{r}_n \in S = R/y_1, \dots, y_s$

all zero-divisors

if the map $S \xrightarrow{\begin{pmatrix} \bar{r}_1 \\ \vdots \\ \bar{r}_n \end{pmatrix}} S^n$

were injective then some S-lin. combo of $\bar{r}_i$'s would be not a zero-div.

have to say "prime avoidance" or "associated prime" unfortunately, so just take it on faith. $\boxtimes$

[NB: without the black box, we proved $s \leq n-k$.]