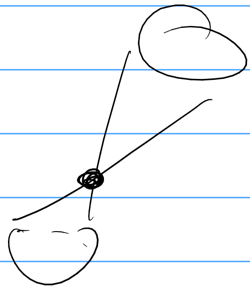
Usual set-up:

R a Noeth. local ring.

$\underline{m}$ = max'l ideal

$k = R/\underline{m}$ residue field.



a while ago:

for a fin. gen module M,

$\dim_k(M \otimes k)$ = min'l # generators for M.

$\mathrm{Tor}_*(M,k)$ tells proj. dim M

proj dim k = glob. dim R

let n = min # gens. of $\underline{m}$

d = depth of $\underline{m}$ = longest regular seq in $\underline{m}$

↳ compute it with a Koszul complex

if $n = d$ then any minimal set of gens

is a regular seq

so the Koszul complex is a free res of $k = R/\underline{m}$

so proj dim $k = n$.

Today, two things:

- proj dim $k \geq n$
- if $n > d$ then proj dim $= \infty$.

Maybe soon: depth$(\underline{m}) \leq \dim R \leq$ # gens of $\underline{m}$

if $\dim R = n$ then $n = d$.

Warm up: if depth$(\underline{m}) = 0$
then either $\#$ gens $(\underline{m}) = 0$
$\rightsquigarrow$ R is a field
$\rightsquigarrow$ glob dim $R = 0$
or glob dim $R = \infty$.

let $r_1, \dots, r_n$ be min'l gens for $\underline{m}$

koszul complex

$$0 \to R \xrightarrow{\begin{pmatrix} r_n \\ -r_{n-1} \\ \vdots \\ \pm r_1 \end{pmatrix}} R^n \to \cdots \to R^{\binom{n}{2}} \to R^n \to R \to 0$$

depth $= 0$, so $H_n(\text{Kosz}) \neq 0$

so $\exists$ non-zero $z \in R$ s.t. $z \cdot r_i = 0 \ \forall i$
so $z \cdot \underline{m} = 0$

(example: $R = k[x,y]/(x^2, y^2)$
$\underline{m} = (x, y)$
could take $z = xy$, but not x or y)

consider

$$\begin{tikzcd} 0 \arrow[r] & \underline{m} \arrow[r] \arrow[dr, "0"'] & R \arrow[r] \arrow[d, "\cdot z"] & k \arrow[r] \arrow[dl, dashed, "\exists"] & 0 \\ & & R & & \end{tikzcd}$$

$\exists$ a non-zero map $k \to R$

has to be injective because k has no submodules

get $\quad 0 \to k \to R \to \text{coker} \to 0$

apply $- \otimes k$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
 & & 0 & & & & \\
\mathrm{Tor}_3(k,k) & \to & 0 & \to & \mathrm{Tor}_3(\text{coker}, k) & \to & \\
\mathrm{Tor}_2(k,k) & \to & 0 & \to & \mathrm{Tor}_2(\text{coker}, k) & \to & \\
\mathrm{Tor}_1(k,k) & \to & 0 & \to & \mathrm{Tor}_1(\text{coker}, k) & \to & \\
k & \to & k & \to & \text{coker} \otimes k & \to & 0
\end{array}$$

if $k \to k$ is an iso then
$\text{coker} \otimes k = 0$ so $\text{coker} = 0$
so $k \cong R$.

if $k \to k$ is zero then
$\mathrm{Tor}_1(\text{coker}, k) \neq 0$
so proj dim coker ≥ 1
so glob dim $R \geq 1$
so $\mathrm{Tor}_i(k,k) \neq 0$ for some $i \geq 1$

$$\mathrm{Tor}_i(k,k) \cong \mathrm{Tor}_{i+1}(\text{coker}, k) \quad \text{by long exact seq.}$$

so glob dim is bigger still.

continue ... glob dim $= \infty$.

(Fact we're using: if $\mathrm{Tor}_{>d}(k,k)=0$
then $\mathrm{glob\,dim}\, R \le d$.)

Now: depth $\underline{m} = d$

choose gens $\underline{m} = (r_1, \dots, r_d, r_{d+1}, \dots, r_n)$

where first d form a reg. seq.

let $S = R/_{r_1, \dots, r_d}$.

Kosz complex of $r_1, \dots, r_d$

gives a free res of S (as an R-mod)

apply $\otimes k$, find $\mathrm{Tor}_i(S,k) = k^{\binom{d}{i}}$

so proj. dim $S = d$.

as before,

$$S \xrightarrow{\begin{pmatrix} r_{d+1} \\ -r_{d+2} \\ \vdots \\ \pm r_n \end{pmatrix}} S^{n-d}$$

is not injective bec. the reg. seq. $r_1 \sim r_d$ was maximal

so $\exists$ an injection $k \hookrightarrow S$:

(because $\exists\, z \in S$ with $z \cdot \bar{r}_i = 0$ $\forall i$

$$\begin{tikzcd} 0 \arrow[r] & (\bar{r}_{d+1} \dots \bar{r}_n) \arrow[r] \arrow[dr, bend right, "0"'] & S \arrow[r] \arrow[d, "\cdot z"] & k \arrow[r] \arrow[dl, dashed, "\exists"] & 0 \\ & & S & & \end{tikzcd}$$
)

consider $0 \to k \to S \to \text{coker} \to 0$

apply $- \otimes k$

$$\to \mathrm{Tor}_{d+2}(k,k) \to 0 \to \mathrm{Tor}_{d+2}$$
$$\to \mathrm{Tor}_{d+1}(k,k) \to 0 \to \mathrm{Tor}_{d+1}(k,\text{coker})$$
$$\to \mathrm{Tor}_d(k,k) \to \underset{\substack{\shortparallel \\ k}}{\mathrm{Tor}_d(k,S)} \to \mathrm{Tor}_d(k,\text{coker})$$

if proj dim $k > d$ then proj dim $k = \infty$.

then some $\mathrm{Tor}_\ell(k,k) \neq 0$ $\ell > d$

so $\mathrm{Tor}_{\ell+1}(\text{coker}, k) \neq 0$

so some later $\mathrm{Tor}(k,k) \neq 0$ and so on.

Why is glob dim R not $<$ depth$(\underline{m})$?

let $r_1 \dots r_n$ be min. gens. of $\underline{m}$

$$\begin{array}{llllllllll}
\text{Koszul} & \cdots \to & R^{\binom{n}{3}} & \to & R^{\binom{n}{2}} & \to & R^n & \to R \to & R/\underline{m} & \to 0 \\
\text{(maybe not exact)} & & \downarrow \exists \varphi_3 & & \downarrow \exists \varphi_2 & & \| & \quad \| & \| & \\
\text{min res:} & \cdots \to & R^{n_3} & \to & R^{n_2} & \to & R^n & \to R \to & R/\underline{m} & \to 0
\end{array}$$

claim: $\varphi_2, \varphi_3, \dots$ are split injective

so $n_2 \geq \binom{n}{2}$, $n_3 \geq \binom{n}{3}$ etc.

so proj dim $R/\underline{m} \geq n$.