Ext and Extensions.

R = any ring

M, N = modules.

an *extension* of M by N is a s.e.s.

$$0 \to N \to E \to M \to 0$$

consider these either up to *iso*:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d, "\cong"] & M \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

or up to *equivalence* which is finer:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "1"] & E \arrow[r] \arrow[d, "\cong"] & M \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Get a map

$$\{\text{extensions up to equiv}\} \longrightarrow \mathrm{Ext}^1_R(M, N)$$

either apply $\mathrm{Hom}(M, -)$ to get

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,E) \to \mathrm{Hom}(M,M) \to \mathrm{Ext}^1(M,N)$$

$$1_M \longmapsto \, ?$$

or apply $\mathrm{Hom}(-,N)$ to get

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(E,N) \to \mathrm{Hom}(N,N) \to \mathrm{Ext}^1(M,N)$$

$$1_N \longmapsto ?$$

Thm: they're the same.
it's a bijection

Also have 2-step extensions

$$0 \to N \to A \to B \to M \to 0$$

$\mathrm{Ext}^2_R(M,N)$ classifies these up to "having a map"

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "1"] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & N \arrow[r] & A' \arrow[r] & B' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

don't require middle maps to be $\cong$.

Ext^3, Ext^4, ...

$0 \in \mathrm{Ext}^n$ corresp to

$$0 \to N \xrightarrow{1} N \to 0 \to \cdots \to 0 \to M \xrightarrow{1} M \to 0$$

product $\mathrm{Ext}^i(M,N) \otimes \mathrm{Ext}^j(N,L) \to \mathrm{Ext}^{i+j}(M,L)$
corresp. to concatenating extensions.

$$0 \to N \to A \to \cdots \to M \to 0$$

$$\|$$

$$0 \to L \to \cdots \to B \to N \to 0$$

Yoneda

Topological digression:

short exact seqs corresp. to fiber bundles

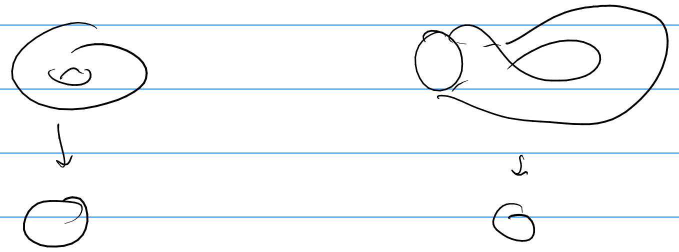
$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \\ & & X \end{array}$$

"$\mathrm{Ext}^1(X, F)$" $\rightsquigarrow$ $H^1(X, \underline{\mathrm{Diff}}(F))$

$$= [X, B\mathrm{Diff}(F)]$$

"Extensions" of S^1 by S^1:

torus Klein bottle



"Extensions" of S^2 by S^1 $\cong \mathbb{Z}$

1 = Hopf fibration

Group Coho: ask about

$$H^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

$$= H^*(BG, A)$$

$\uparrow$ local system

Examples

① $\mathrm{Ext}^1(\mathbb{Z}/5, \mathbb{Z}/5)$

$$0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to \mathbb{Z}/5 \to 0$$

$\mathrm{Hom}(-, \mathbb{Z}/5)$

$$\mathbb{Z}/5 \xleftarrow{5=0} \mathbb{Z}/5$$

$\mathrm{Ext}^1 = \mathbb{Z}/5 = \{0, 1, 2, 3, 4\}$ orbits as $\mathrm{Aut}(\mathbb{Z}/5)$ acts.

0: the split one

$$0 \to \mathbb{Z}/5 \to \mathbb{Z}/5 \oplus \mathbb{Z}/5 \to \mathbb{Z}/5$$

the non-split ones

1: $0 \to \mathbb{Z}/5 \to \mathbb{Z}/25 \to \mathbb{Z}/5 \to 0$

 $1 \mapsto 1$ (for $\mathbb{Z}/25 \to \mathbb{Z}/5$)

 $1 \mapsto 5$

2: $1 \mapsto 10$

3: $1 \mapsto 15$

4: $1 \mapsto 20$

all iso but not equivalent.

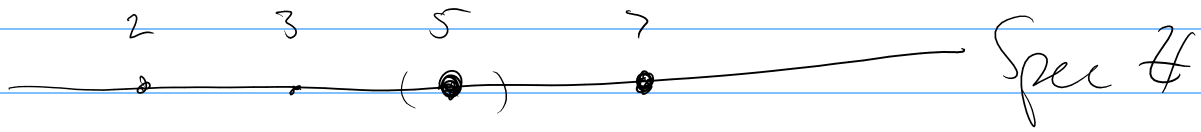
Reason: $\underset{\mathbb{Z}/4}{\underset{\shortparallel}{\mathrm{Aut}(\mathbb{Z}/5)}}$ acts on $\mathrm{Ext}^1(\mathbb{Z}/5, \mathbb{Z}/5)$

② $\operatorname{Ext}^1(\mathbb{Z}/5, \mathbb{Z}/7)$

$$\mathbb{Z}/7 \xleftarrow{5} \mathbb{Z}/7$$

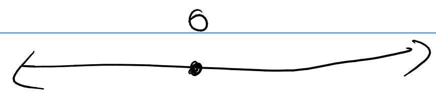
$\operatorname{Ext}^1 = 0$

only extension is $\mathbb{Z}/5 \oplus \mathbb{Z}/7$



③ Similar: $R = k[x]$

$\operatorname{Ext}^1(R/x, R/x)$



$$0 \to R \xrightarrow{x} R \to R/x \to 0$$

$\operatorname{Hom}(-, R/x)$

$$R/x \xleftarrow{0} R/x$$

$\operatorname{Ext}^1 = R/x \cong k$

$\operatorname{Aut}(R/x) = k^*$ acts.

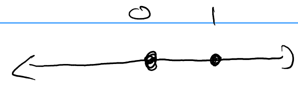
two orbits: 0 and not 0

two extensions: the split one and

$$0 \to R/x \xrightarrow{x} R/x^2 \to R/x \to 0$$

$2x, 3x, 4x$

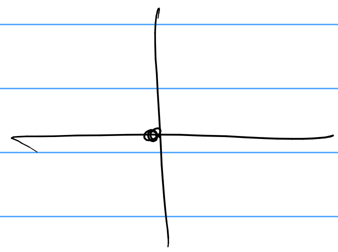
OTOH, $\mathrm{Ext}^1(R/x, R/x-1) = 0$



$$R/x-1 \xleftarrow{1} R/x-1$$

(4) $R = k[x,y]$

$$\mathrm{Ext}^1(R/(x,y), R/(x,y))$$



Koszul res:

$$0 \to R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{(x\ y)} R \to R/(x,y) \to 0$$

do $\mathrm{Hom}(-, R/(x,y))$ ... differentials become 0

$$\mathrm{Ext}^1 = \left(R/(x,y)\right)^2 = k^2$$

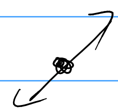
many non-iso exts:

$$0 \to R/(x,y) \to R/(x^2,y) \to R/(x,y) \to 0$$

$$0 \to R/(x,y) \to R/(x,y^2) \to R/(x,y) \to 0$$

$$0 \to R/(x,y) \to R/(y-x, x^2) \to R/(x,y) \to 0$$

$$= (y-x, y^2)$$



Non-split extensions

up to iso rather than equiv:

$$k^2 \setminus 0 \Big/ k^* = \mathrm{Aut}\left(R/_{(x,y)}\right) = \mathbb{P}^1_k$$

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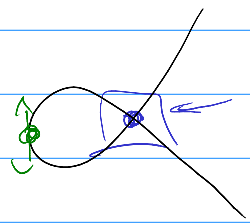
fun: $R = k[x,y] \Big/ y^2 - x^2 - x^3$



$\mathrm{Ext}^1(k,k) = k^2$

for $k = R/_{(x,y)}$

$\mathrm{Ext}^1(k,k) = k^1$

for $k = R/_{(x+1,y)}$

worksheet: beyond self-extensions.