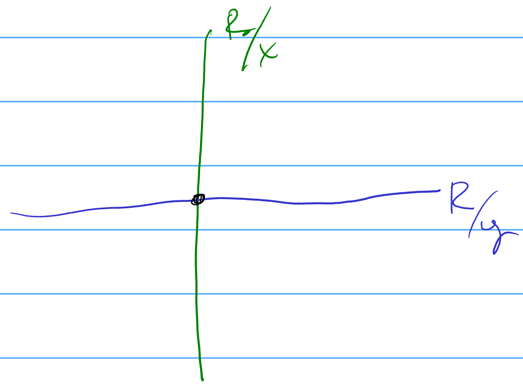
Worksheet from Friday:

over $R = k[x,y]$, computed

$$\mathrm{Ext}^1(R/x, R/y) = R/(x,y) \cong k$$



gives extensions up to equiv.

up to iso, mod out $\mathrm{Aut}(R/x)$ and $\mathrm{Aut}(R/y)$:

$$\mathrm{End}(R/x) = \mathrm{Hom}(R/x, R/x) = R/x = k[y]$$

as a ring!

$$\mathrm{Aut} = k[y]^* = k^*$$

k/k^* = two possibilities

$R/x \oplus R/y$ and R/xy

rank of this $\otimes R/\underline{m}$ as $\underline{m}$ varies

is 0 over $k^2 \setminus +$

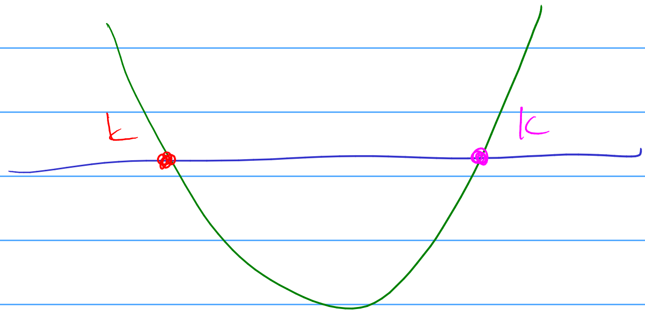
1 over $+$

2 over origin

just 0 and 1.

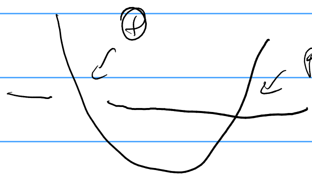
more interesting:

$$\operatorname{Ext}^1\left(R/y,\ R/x^2-1-y\right) = R/(x+1,y) \oplus R/(x-1,y)$$



two numbers: if $(0,0)$ $\rightsquigarrow$ direct sum

if $(0, \neq 0)$ $\rightsquigarrow$



proj. of rk 1

if $(\neq 0, \neq 0)$ $\rightsquigarrow$ interesting line bundle (proj module)

over $R/y(x^2-1-y)$

$(1,1)$ $\rightsquigarrow$ free module

$(1,2)$ $\rightsquigarrow$ some other proj. module?

up to Aut of either module $= k^*$,

only care about ratio of two numbers.

$$(a,b) \in R/_{(x+1,\,y)} \oplus R/_{(x-1,\,y)} = k^2$$

$$\ker \hookrightarrow R/_{y} \oplus R/_{x^2-1-y} \xrightarrow{\begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}} k \oplus k$$

maybe

Spectral Sequences.

<u>motivation</u>: we saw that if $R \to S$ is flat

then $\mathrm{Tor}_i^R(M,N) \otimes S \simeq \mathrm{Tor}_i^S(M\otimes S,\ N\otimes S)$

what if $R \to S$ is not flat?

<u>simpler</u>: $R \longrightarrow S$ a ring map, not flat

M a right R-mod

N a left S-module

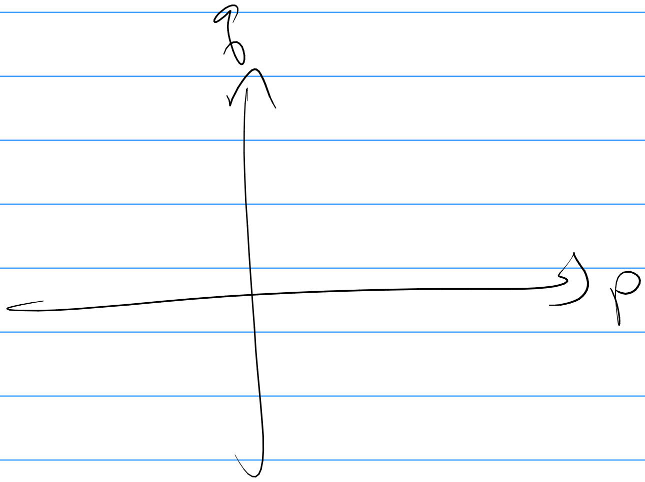
then $\exists$ a 3^{rd} quadrant spectral seq.

$$E_2^{p,q} = \mathrm{Tor}^S_{-p}\left(\mathrm{Tor}^R_{-q}(M,S),\ N\right) \Rightarrow \mathrm{Tor}^R_{-p-q}(M,N)$$

think about

$$\begin{tikzcd} \text{mod-}R \arrow[r, "-\otimes_R S"] \arrow[rr, bend right, "-\otimes_R N"'] & \text{mod-}S \arrow[r, "-\otimes_S N"] & \text{Ab. groups} \end{tikzcd}$$

a spectral seq: a grid



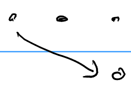
a sequence of such grids ("pages")

$E_0^{p,q}$, $E_1^{p,q}$, $E_2^{p,q}$, $E_3^{p,q}$ ...

each has some differentials with $d^2=0$

E_0 

E_1

E_2 

E_3 

etc

Homology of each page gives the next page.

hope: eventually all the d's have to be 0

(often: they're all 0 from E_1 or E_2)

→ then on the "E_∞" page,

look at the diagonals

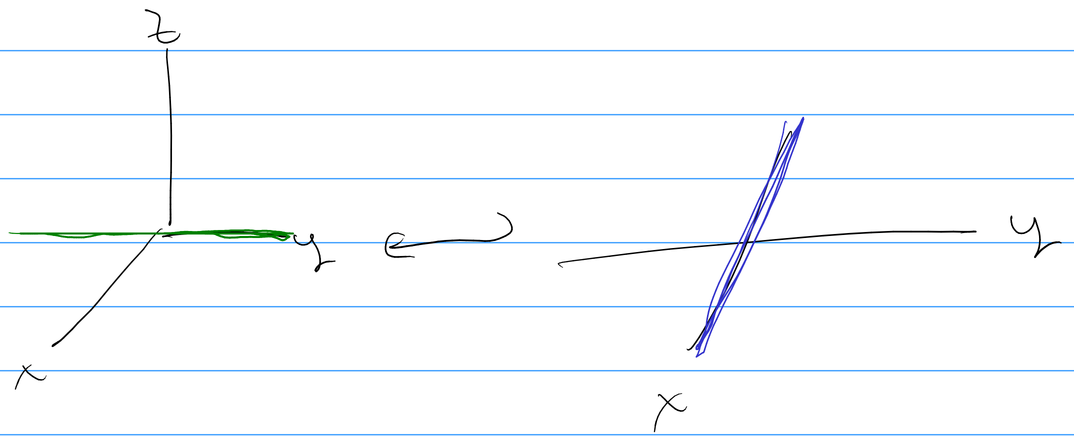
the n^{th} thing you're computing has a filtration whose quotients are

the things on the n^{th} diagonal: $E_\infty^{p,n-p}$ ✓

$$E_2^{p,q} = \mathrm{Tor}^S_{-p}(\mathrm{Tor}^R_{-q}(M,S), N) \Rightarrow \mathrm{Tor}^R_{-p-q}(M,N)$$

example: $R = k[x,y,z] \longrightarrow S = R/z = k[x,y]$

$M = R/x,z$ $\qquad N = S/y$



on the first day, we were surprised at $\mathrm{Tor}^R_1(M,N)$, because they intersect transversely in the xy-plane, just not in space.

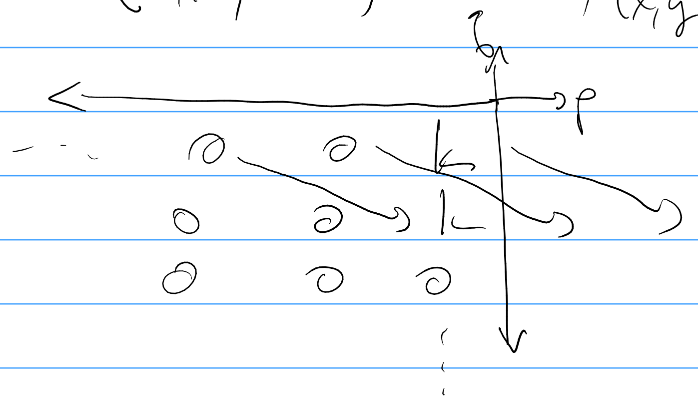
Compute: $\mathrm{Tor}_*^R(M,S) = M, M, 0 \dots$ but as an S-module, should call it S/x, not M.

$0 \to R \xrightarrow{x} R \to S \to 0$

$M \xrightarrow{0} M$

$\mathrm{Tor}_*^S(S/x, N) = \overset{=k}{S/(x,y)}, 0, 0, 0 \dots$

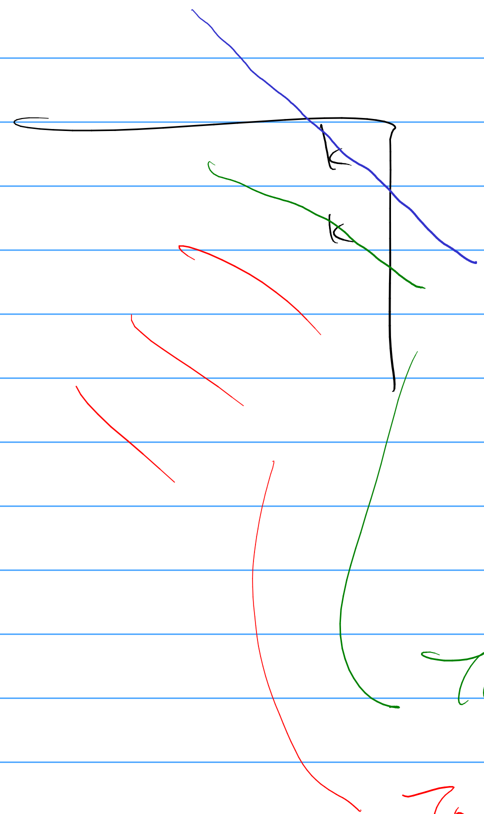
E_2:



differentials have nowhere to go.

$E_2 = E_3 = \dots = E_\infty$

E_∞:



$\mathrm{Tor}_0^R(M,N)$ has a filt whose quots are $\dots, 0, k, 0, \dots$

So $\mathrm{Tor}_0^R(M,N) = k$

$\mathrm{Tor}_1^R(M,N) = k$ as well.

$\mathrm{Tor}_{\geq 2}^R(M,N) = 0$.