

Spectral Sequences Cont'd.

maybe read Eisenbud A3.13
or Weibel ch. 5, esp. § 5.6

Change-of-rings s.s.

$$R \rightarrow S$$

M an R -module

N an S -module

$$E_2^{p,q} = \text{Tor}_-^S(\text{Tor}_-^R(M, S), N) \Rightarrow \text{Tor}_-^R(M, N)$$

underlying undervived fact:

$$(M \otimes_R S) \otimes_S N = M \otimes_R N$$

$$\text{Spec } R \xleftarrow{f} \text{Spec } S$$

\tilde{M} \tilde{N}

projection formula: $f_* (f^* \tilde{M} \otimes \tilde{N}) = \tilde{M} \otimes f_* \tilde{N}$

similar formulas exist in cohomology...

$$f_* \alpha \cap \varphi = f_* (\alpha \cap f^* \varphi) \quad \text{Hatcher}$$

and in calculus:

p. 241

$$\int \underset{\substack{\uparrow \\ \text{like } \tilde{N}}}{g(x,y)} \underset{\substack{\uparrow \\ \text{like } \tilde{M}}}{h(x)} dy = h(x) \int \underset{\substack{\uparrow \\ \text{like } f_*}}{g(x,y)} dy$$

∃ other change-of-rings spectral seqs.

$$\text{know } \text{Hom}_S(N, \text{Hom}_R(S, M)) = \text{Hom}_R(N, M)$$

$$\text{so } E_2^{P, q} = \text{Ext}_S^p(N, \text{Ext}_R^q(S, M)) \Rightarrow \text{Ext}_R^{p+q}(N, M)$$

↳ first quadrant

$$\text{and } \text{Hom}_S(M \otimes_R S, N) = \text{Hom}_R(M, N)$$

$$\text{so } E_2^{P, q} = \text{Ext}_S^p(\text{Tor}_f^R(M, S), N) \Rightarrow \text{Ext}_R^{p+q}(M, N)$$

↳ also first quadrant

Worksheet: $R = k[x, y] \rightarrow S = k[x, y]_{(x, y)}$

$$M = R/x$$

$$N = S/y (= R/y)$$

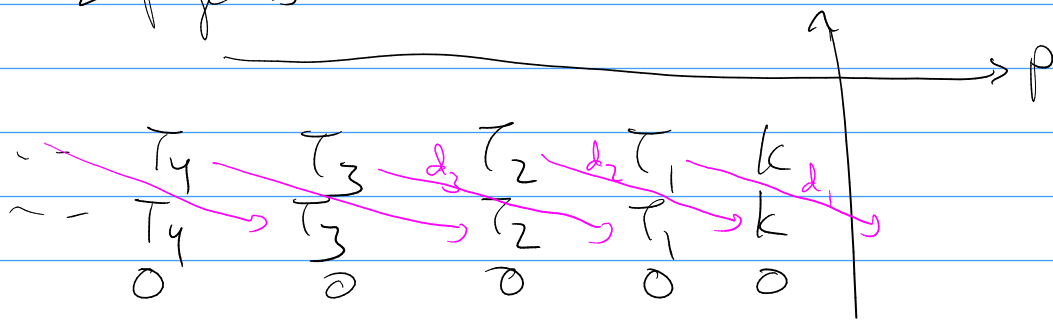
$$\text{Tor}_+^R(M, N) = R/(x, y), 0, 0, \dots$$

$$\text{Tor}_+^R(M, S) = S/x, S/x, 0, \dots$$

by hand $\text{Tor}_+^S(S/x, N) = k, 0, k, 0, k, 0, \dots$
 but could have gotten this from first two
 + spectral seq:

$$\text{Tor}_p^S(\text{Tor}_f^R(M, S), N) \Rightarrow \text{Tor}_{-p}^R(M, N)$$

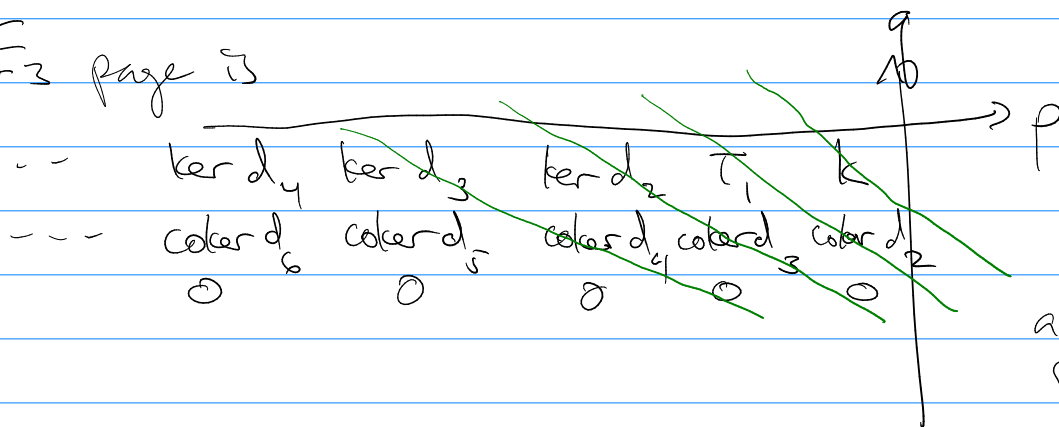
\mathbb{F}_2 page is



$$\begin{cases} \text{Tor}_0^R(M, S) = S/x \\ \text{Tor}_1^R(M, S) = S/x \\ \text{Tor}_2 = 0 \end{cases}$$

let T_i be short for $\text{Tor}_i^S(S/x, S/y)$
 know $T_0 = S/(x, y) = k$

\mathbb{F}_3 page is



differentials are all off the page, so $\mathbb{F}_3 = \mathbb{F}_\infty$.

from 0th diag: $\text{Tor}_0^R(M, N) = R/(x, y) = k$ has a filt.
 whose quot. are $\dots, 0, k, 0, \dots$
 (already knew that)

from -1st diag: $\text{Tor}_1^R(M, N) = 0$ has a filt.
 whose quot. are T_1 and $\text{coker } d_2$

so $T_1 = 0$ and d_2 is surjective

-2nd diag: $\text{Tor}_2^R(M, N) = 0$ vs $\text{ker } d_2 = 0$ and $\text{coker } d_3 = 0$

so d_2 is inj. and d_3 is surj.

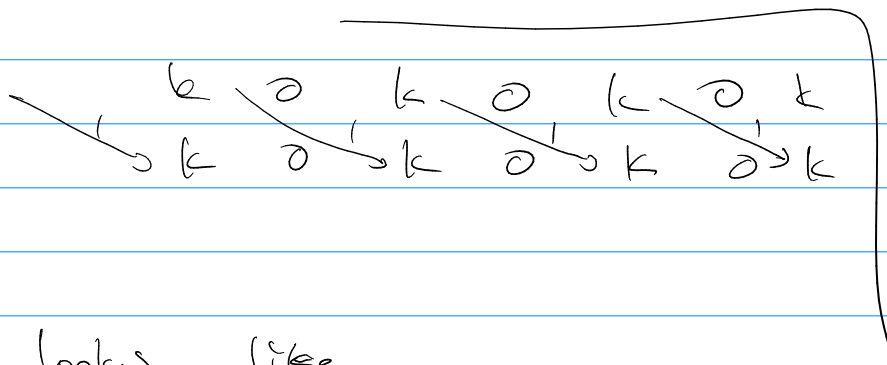
so $T_0 = k$ $T_1 = 0$

$d_2: T_2 \rightarrow T_0$ is an iso, so $T_2 = k$

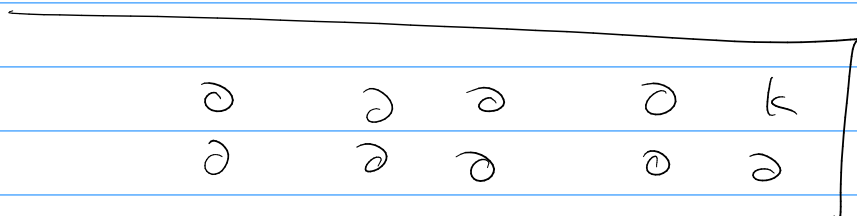
$d_3: T_3 \rightarrow T_1$ is an iso, so $T_3 = 0$

sim. $T_4 = k, T_5 = 0, \dots$

E_2 page had to look like



E_3 looks like



Application of change-of-rings s.s.

let $R =$ Noether comm ring
 $f \in R$ not a zero-div.

$S = R/f$ N a fin gen S -mod.

if $\text{proj dim}_S N = d < \infty$

then $\text{proj dim}_R N = d+1$

just saw a counterexample if $d = \infty$,
with $S = k[x,y]_{xy}$ $N = S/y$
 $\text{proj dim}_S N = \infty$ but $\text{proj dim}_R (N = R/y) = 1$

apply the s.s. with $M = R/\underline{m}$ for any maximal ideal \underline{m}
 $\text{proj dim}_R(N) = \sup_{\underline{m}} \sup \{ i : \text{Tor}_i^R(R/\underline{m}, N) \neq 0 \}$

and same with S .

$\text{Spec } R \longleftrightarrow \text{Spec } S$

max'l ideals of $S = R/f \longleftrightarrow$ max'l ideals of R containing f

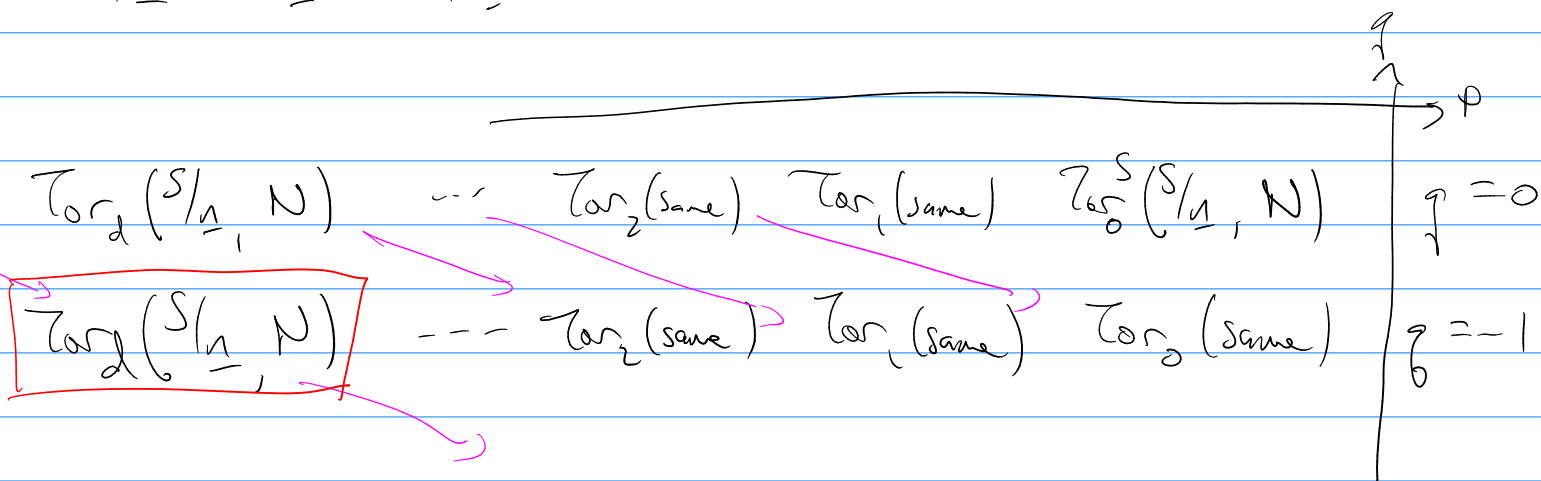
if $f \notin \underline{m}$ then $\text{Tor}_+^R(R/\underline{m}, N) = 0$

$$E_2^{p,q} = \text{Tor}_{-p}^S(\text{Tor}_{-q}^R(R/\underline{m}, S), N) \implies \text{Tor}_{-p-q}^S(R/\underline{m}, N)$$

from $0 \rightarrow R \xrightarrow{f} R \rightarrow S \rightarrow 0$

get $\text{Tor}_a^R(R/\underline{m}, S) = R/\underline{m}, R/\underline{m}, 0, \dots$
if $f \in \underline{m}$

$R/\underline{m} = S/\underline{m}S =: S/\underline{n}$ if $f \notin \underline{m}$ $0, 0, 0, \dots$



\exists some $\underline{n} \subset S \rightsquigarrow$ some $\underline{m} \subset R$

that knows $\text{proj dim}_S(N) = d$

so $\text{Tor}_d^S(S/\underline{n}, N) \neq 0$

no differentials to kill the one in lower left

so it survives to the E_∞ page

and gives $\text{Tor}_{d+1}^R(R/\underline{m}, N) \neq 0$

so $\text{proj dim}_R(N) = d+1$

