# Spectral Sequences Cont'd.

maybe read Eisenbud A3.13
or Weibel ch. 5, esp. §5.6

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Change-of-rings s.s.

$R \longrightarrow S$

$M$ an $R$-module

$N$ an $S$-module

$$E_2^{p,q} = \mathrm{Tor}^S_{-p}\left(\mathrm{Tor}^R_{-q}(M, S), N\right) \Rightarrow \mathrm{Tor}^R_{-p-q}(M, N)$$

underlying underived fact:

$$(M \otimes_R S) \otimes_S N = M \otimes_R N$$

$$\underset{\tilde{M}}{\mathrm{Spec}\, R} \xleftarrow{f} \underset{\tilde{N}}{\mathrm{Spec}\, S}$$

projection formula: $f_*(f^*\tilde{M} \otimes \tilde{N}) = \tilde{M} \otimes f_*\tilde{N}$

similar fmlas exist in cohomology...

$$f_*\alpha \frown \varphi = f_*(\alpha \frown f^*\varphi)$$ Hatcher p. 241

and in calculus:

$$\int \underset{\substack{\uparrow \\ \text{like } \tilde{N}}}{g(x,y)}\, \underset{\substack{\uparrow \\ \text{like } \tilde{M}}}{h(x)}\, dy = h(x) \int g(x,y)\, dy$$

$\int$ like $f_*$

$\exists$ other change-of-rings spectral seqs.

know $\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) = \mathrm{Hom}_R(N, M)$

so $E_2^{p,q} = \mathrm{Ext}_S^p(N, \mathrm{Ext}_R^q(S, M)) \Rightarrow \mathrm{Ext}_R^{p+q}(N, M)$

↳ first quadrant

and $\mathrm{Hom}_S(M \otimes_R S, N) = \mathrm{Hom}_R(M, N)$

so $E_2^{p,q} = \mathrm{Ext}_S^p(\mathrm{Tor}_q^R(M, S), N) \Rightarrow \mathrm{Ext}_R^{p+q}(M, N)$

↳ also first quadrant

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Worksheet: $R = k[x,y] \longrightarrow S = k[x,y]/xy$

$M = R/x \qquad N = S/y \quad (= R/y)$

$\mathrm{Tor}_*^R(M, N) = \underbrace{R/(x,y)}_{=k}, 0, 0 \ldots$

$\mathrm{Tor}_*^R(M, S) = S/x, \; S/x, \; 0, \ldots$

by hand $\mathrm{Tor}_*^S(S/x, N) = k, 0, k, 0, k, 0, \ldots$

but could have gotten this from first two + spectral seq:

$\mathrm{Tor}_{-p}^S(\mathrm{Tor}_{-q}^R(M, S), N) \Rightarrow \mathrm{Tor}_{-p-q}^R(M, N)$

$E_2$ page is

$$\begin{array}{cccccc|}
 & & & & & q \uparrow \quad \longrightarrow p \\
\cdots & T_4 & T_3 & T_2 & T_1 & k \\
\cdots & T_4 & T_3 & T_2 & T_1 & k \\
 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$d_3$, $d_2$, $d_1$

$$\left\{ \begin{array}{l} Tor_0^R(M,S) = S/x \\ Tor_1^R(M,S) = S/x \\ Tor_2 = 0 \end{array} \right.$$

let $T_i$ be short for $Tor_i^S(S/x, S/y)$

know $T_0 = S/_{(x,y)} = k$

$E_3$ page is

$$\begin{array}{cccccc|}
 & & & & & q \uparrow \quad \longrightarrow p \\
\cdots & \ker d_4 & \ker d_3 & \ker d_2 & T_1 & k \\
\cdots & \operatorname{coker} d_6 & \operatorname{coker} d_5 & \operatorname{coker} d_4 & \operatorname{coker} d_3 & \operatorname{coker} d_2 \\
 & 0 & 0 & 0 & 0 & 0
\end{array}$$

differentials are all off the page, so $E_3 = E_\infty$.

from $0^{th}$ diag: $Tor_0^R(M,N) = R/_{(x,y)} = k$ has a filt. whose quots. are $\cdots 0, k, 0 \cdots$ (already knew that)

from $-1^{st}$ diag: $Tor_1^R(M,N) = 0$ has a filt. whose quots are $T_1$ and $\operatorname{coker} d_2$

so $T_1 = 0$ and $d_2$ is surjective

$-2^{nd}$ diag: $Tor_2^R(M,N) = 0 \rightsquigarrow \ker d_2 = 0$ and $\operatorname{coker} d_3 = 0$

So $d_2$ is inj. and $d_3$ is surj.

so $T_0 = k$ $\quad T_1 = 0$

$d_2 : T_2 \to T_0$ is an iso, so $T_2 = k$

$d_3 : T_3 \to T_1$ is an iso, so $T_3 = 0$

sim. $T_4 = k$, $T_5 = 0$, etc.

$E_2$ page had to look like

$$\begin{array}{ccccccc} k & 0 & k & 0 & k & 0 & k \\ k & 0 & k & 0 & k & 0 & k \end{array}$$

$E_3$ looks like

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

Application of change-of-rings s.s.

let $R$ = Noeth comm ring
$f \in R$ not a zero-div.

$S = R/f$ $\qquad$ $N$ a fin gen $S$-mod.

if $\text{proj dim}_S N = d < \infty$

then $\text{proj. dim}_R N = d+1$

[just saw a counterexample if $d = \infty$,
with $S = k[x,y]/xy$ $\quad N = S/y$

$\text{proj dim}_S N = \infty$ but $\text{proj dim}_R (N = R/y) = 1$.]

apply the s.s. with $M = R/\underline{m}$ for any maximal ideal $\underline{m}$

$\text{proj dim}_R(N) = \sup_{\underline{m}} \sup \{ i : \text{Tor}_i^R(R/\underline{m}, N) \neq 0 \}$

and same with $S$...

$\text{Spec } R \hookleftarrow \text{Spec } S$

max'l ideals of $S = R/f$ $\longleftrightarrow$ max'l ideals of $R$ containing $f$

if $f \notin \underline{m}$ then $\text{Tor}_*^R(R/\underline{m}, N) = 0$

$$E_2^{p,q} = \mathrm{Tor}^S_{-p}\big(\underbrace{\mathrm{Tor}^R_{-q}(R/\underline{m}, S)}, N\big) \Longrightarrow \mathrm{Tor}^S_{-p-q}(R/\underline{m}, N)$$

from $0 \to R \xrightarrow{f} R \to S \to 0$

get $\mathrm{Tor}_*(R/\underline{m}, S) = R/\underline{m},\ R/\underline{m},\ 0 \ldots$

if $f \in \underline{m}$

$0,\ 0,\ 0 \ldots$

$R/\underline{m} = S/\underline{m}S =: S/\underline{n}$ if $f \notin \underline{m}$

$$\begin{array}{ccccc|l}
 & & & & & q \\
\mathrm{Tor}_d(S/\underline{n}, N) & \cdots & \mathrm{Tor}_2(\text{same}) & \mathrm{Tor}_1(\text{same}) & \mathrm{Tor}^S_0(S/\underline{n}, N) & q = 0 \\
\boxed{\mathrm{Tor}_d(S/\underline{n}, N)} & \cdots & \mathrm{Tor}_2(\text{same}) & \mathrm{Tor}_1(\text{same}) & \mathrm{Tor}_0(\text{same}) & q = -1
\end{array} \to p$$

$\exists$ some $\underline{n} \subset S \rightsquigarrow$ some $\underline{m} \subset R$

that knows $\mathrm{proj\,dim}_S(N) = d$

so $\mathrm{Tor}^S_d(S/\underline{n}, N) \neq 0$

no differentials to kill the one in lower left

So it survives to the $E_\infty$ page

and gives $\mathrm{Tor}^R_{d+1}(R/\underline{m}, N) \neq 0$

so $\mathrm{proj\,dim}_R(N) = d+1$

could have proved this by resolving $N$ with free $S$-modules

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & \ker \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[rr] \arrow[dr] & & S^{n_2} \arrow[rr] \arrow[ur] & & S^{n_1} \arrow[rr] \arrow[dr] & & S^{n_0} \arrow[r] & N \arrow[r] & 0 \\
 & \ker \arrow[ur] \arrow[dr] & & & & \ker \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

and doing long exact seq for

$- \otimes R/\underline{m}$ applied to

$$0 \to \ker \to S^{n_i} \to \ker \to 0$$

and then a big induction

on $\mathrm{proj\ dim}_S(N)$.

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next time: spectral seq of a double cx.

balancedness of Tor and Ext.