Where are these spectral seqs coming from?

Def a <u>double complex</u> $C^{p,q}$ of R-modules has a horizontal diff $d_{horz} : C^{p,q} \to C^{p+1,q}$ and a vertical diff $d_{vert} : C^{p,q} \to C^{p,q+1}$

they commute, and both have $d^2 = 0$

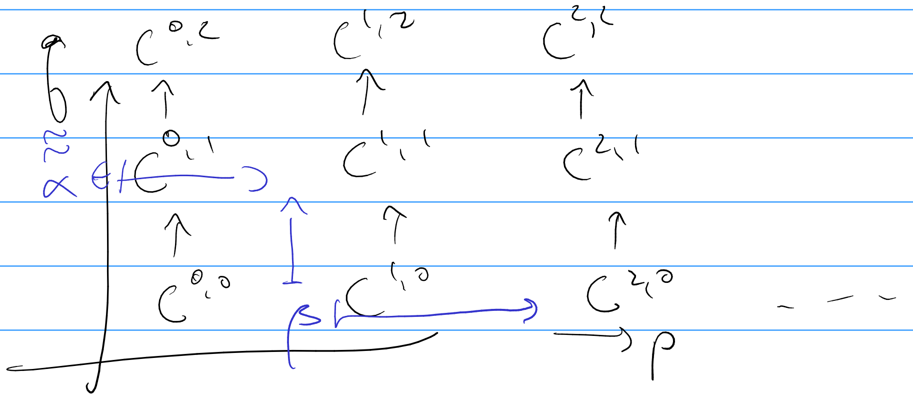
the <u>total complex</u> $T = tot(C^{p,q})$ is given by $T^k = \bigoplus_{p+q=k} C^{p,q}$

and $d = d_{vert} + (-1)^q d_{horz}$

it satisfies $d^2 = 0$.

From a double complex, get a spectral seq.

$E_0^{p,q} = C^{p,q}$ $\quad d_0 = d_{vert}$

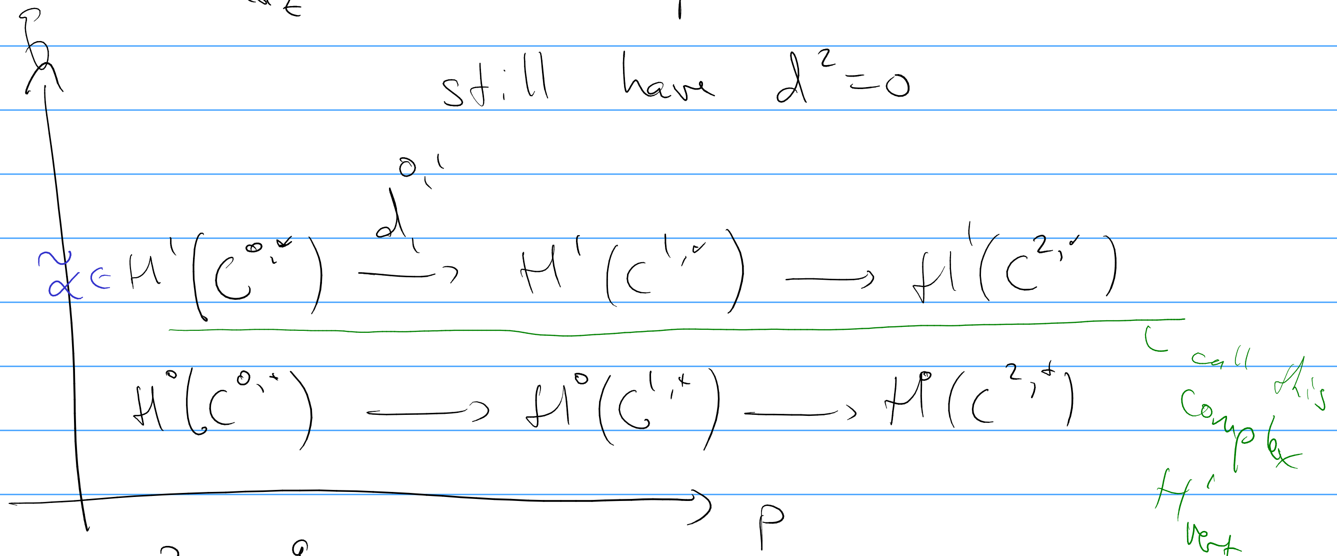


$E_1^{p,q} = H^q_{vert}(C^{p,*})$

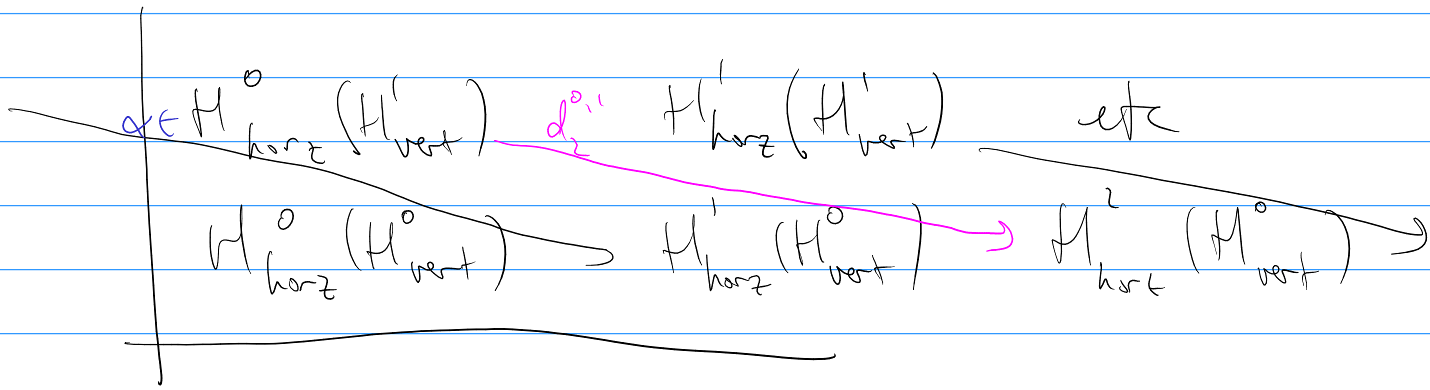
think of $C^{0,*}$, $C^{1,*}$, $C^{2,*}$ as complexes, d_{horz} as maps of complexes

so d_{horz} induces maps on H^*

still have $d^2 = 0$



$E_2^{p,q} = H^p_{horz} H^q_{vert}$



what does d_2 look like?

given some $\alpha \in H^0_{horz}(H^1_{vert})$,

lift to $\tilde{\alpha} \in H^1(C^{0,*})$

know that $d_1^{0,1}(\tilde{\alpha}) = 0 \in H^1(C^{1,*})$

lift further to $\tilde{\tilde{\alpha}} \in C^{0,1}$

so $d_{horz}(\tilde{\alpha}) \in C^{1,1}$

and it's zero in $H^1(C^{1,*})$

so $\exists\, \beta \in C^{1,0}$ such that

$$d_{vert}\beta = d_{horz}\tilde{\alpha}$$

now define $d_2^{1,0}(\alpha) = [d_{horz}(\beta)] \in H^2_{horz}(H^0_{vert})$

check: it's well-defined.

$$(d_2)^2 = 0$$

<u>Thm</u> if $C^{p,q}$ lives in the first or third quadrant then

spec. seq $\Rightarrow H^{p+q}$(total complex).

<u>rmk</u>: can also transpose $C^{p,q}$
$tot(C^{q,p}) \cong tot(C^{p,q})$
but the spectral sequence of the transpose gives different info.

Application: change-of-rings s.s. that we've been using

$$R \longrightarrow S$$

M an R-mod

N an S-mod

$$E_2^{p,q} = \operatorname{Tor}^S_{-p}\left(\operatorname{Tor}^R_{-q}(M,S),\ N\right) \Rightarrow \operatorname{Tor}^R_{-p-q}(M,N)$$

To set it up, first resolve S by free R-modules.

$$\cdots \to R^{m_2} \longrightarrow R^{m_1} \longrightarrow R^{m_0} \to S \to 0$$

then resolve N by free S-modules

$$\cdots \to S^{n_2} \longrightarrow S^{n_1} \longrightarrow S^{n_0} \to N \to 0$$

$$\begin{array}{ccccccc}
\cdots \to & R^{m_0 n_2} & \longrightarrow & R^{m_0 n_1} & \longrightarrow & R^{m_0 n_0} \\
 & \uparrow & & \uparrow & & \uparrow \\
\cdots \to & R^{m_1 n_2} & \longrightarrow & R^{m_1 n_1} & \longrightarrow & R^{m_1 n_0} \\
 & \uparrow & & \uparrow & & \uparrow \\
\cdots \to & R^{m_2 n_2} & \longrightarrow & R^{m_2 n_1} & \longrightarrow & R^{m_2 n_0} \\
 & \uparrow & & \uparrow & & \uparrow \\
 & \vdots & & \vdots & & \vdots
\end{array}$$

this is my double complex

$$C^{p,q} = R^{m_{-q} n_{-p}} \text{ with those differentials}$$

first claim: $T := \operatorname{tot}(C^{p,q})$ is a res of N by free R-modules.

just run the s.s. on this double cx.

$E_0^{p,q}$ is above.

$E_1^{p,q}$ is
$$\begin{array}{ccccc|} S^{n_2} & \to & S^{n_1} & \to & S^{n_0} \\ 0 & & 0 & & 0 \\ 0 & & 0 & & 0 \end{array}$$

$E_2^{p,q}$ is
$$\begin{array}{ccc|} 0 & 0 & N \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

so T is a complex of free R-mods with $H^0 = N$, otherwise exact.

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now consider the double complex $M \otimes C^{p,q}$

$E_0^{p,q}$ is $M \otimes R^{m_{-q} n_{-p}} = M^{m_{-q} n_{-p}}$

$E_1^{p,q}$ is $\mathrm{Tor}_{-q}^R(M, S)^{n_{-p}}$

rows are $\mathrm{Tor}_{-q}^R(M,S) \otimes$ free res of N

So $E_2^{p,q} = H^p_{horz} H^q_{vert} = \mathrm{Tor}^S_{-p}\left(\mathrm{Tor}^R_{-q}(M,S), N\right)$

end of the s.s. is $H^{p+q}(M \otimes T)$

which is $\mathrm{Tor}^R_{-p-q}(M, N)$

because T was a res of N by free R-mods.